Chapter 3

LINEAR PROGRAMMING

Introduction

Linear programming is the most widely applied of all of the optimization methods. The technique has been used for optimizing many diverse applications, including refineries and chemical plants, livestock feed blending, routing of aircraft, and scheduling their crews. Many industrial allocation and transportation problems can be optimized with this method. The application of linear programming has been successful, particularly in cases of selecting the best set of values of the variables when a large number of interrelated choices exist. Often such problems involve a small improvement per unit of material flow time's large production rates to have as the net result be a significant increase in the profit of the plant. A typical example is a large oil refinery where the stream flow rates are very large, and a small improvement per unit of product is multiplied by a very large number to obtain a significant increase in profit for the refinery.

The term *programming* of linear programming does not refer to computer programming but to scheduling. Linear programming was developed about 1947, before the advent of the computer, when George B. Dantzig (1) recognized a generalization in the mathematics of scheduling and planning problems. Developments in linear programming have followed advances in digital computing, and now problems involving several thousand independent variables and constraints equations can be solved.

In this chapter a geometric representation and solution of a simple linear programming problem will be given initially to introduce the subject and illustrate the way to capitalize on the mathematical structure of the problem. This will be followed by a presentation of the simplex algorithm for the solution of linear programming problems. Having established the computational algorithm, we will give the procedure to convert a process flow diagram into a linear programming problem, using a simple petroleum refinery as an illustration. The method of solution, using large linear programming computer codes, then will be described, and the solution of the refinery problem using the IBM Mathematical Programming System Extended (MPSX), will illustrate the procedure and give typical results obtained from these large codes. Once the optimal solution has been obtained, sensitivity analysis procedures will be detailed which use the optimal solution to determine ranges on the important parameters where the optimal solution remains optimal. Thus, another linear programming solution is not required. This will be illustrated also using results of the refinery problem obtained from the MPSX solution. Finally, a summary will be given of extensions to linear programming and other related topics.

Concepts and Geometric Interpretation

As the name indicates, all of the equations that are used in linear programming must be linear. Although this appears to be a severe restriction, there are many problems that can be cast in this context. In a linear programming formulation, the equation that determines the profit or cost of operation is referred to as the *objective function*. It must have the form of the sum of linear terms. The equations that describe the limitations under which the system must operate are called the *constraints*. The variables must be nonnegative, i.e., positive or zero only.

The best way to introduce the subject is with an example. This will give some geometric intuition about the mathematical structure of the problem and the way this structure can be used to find an optimal solution.

Example 3.1

A chemical company makes two types of small solid fuel rocket motors for testing; for motor A the profit is \$3.00 per motor and for motor B the profit is \$4.00 per motor. A total processing time of 80 hours per week is available to produce both motors. An average of four hours per motor is required for A, but only two hours per motor is required for B. However, due to hazardous nature of the material in B, a preparation time of five hours is required per motor, and a preparation time of two hours per motor is required for A. The total preparation time of 120 hours per week is available to produce both motors. Determine the number of each motor that should be produced to maximize the profit.

Solution: The objective function and constraint equations for this case are:

maximize: $3A + 4B$ Profit

subject to: $4A + 2B \leq 80$ Processing Time

$2A + 5B \leq 12$ Preparation Time

$A, B \geq 0$

It would be tempting to make all B motors using the preparation time limitation 120/5 = 24 for a profit of \$96. If all A motors were made, there is a processing time limitation 80/4 = 20 for a profit of \$60. However, there is a best solution, and this can be seen from Figure 3-1. The small arrows show the region enclosed by the constraint equations that is feasible for the variables. For the processing time and preparation time, any values of the variables lying above the lines violate the constraint equations. Consequently, feasible values must lie on or inside the lines, and the *A* and *B* axes (since *A* and *B* must be nonnegative). This is called the *feasible region*. The objective function is shown in Figure 3.1 for *P* = 96, and this is the one of the family of lines:

$$3A + 4B = P$$

or

$$A = -\ (4/3)\ B + P/3$$

where *P* can increase as long as the values of the variables *A* and *B* stay in the feasible region. By increasing *P*, the profit equation shown above moves up with a constant slope of - 4/3, and *P* reaches the maximum value in the feasible region at the vertex *A* = 10, *B* = 20, where *P* = \$110.

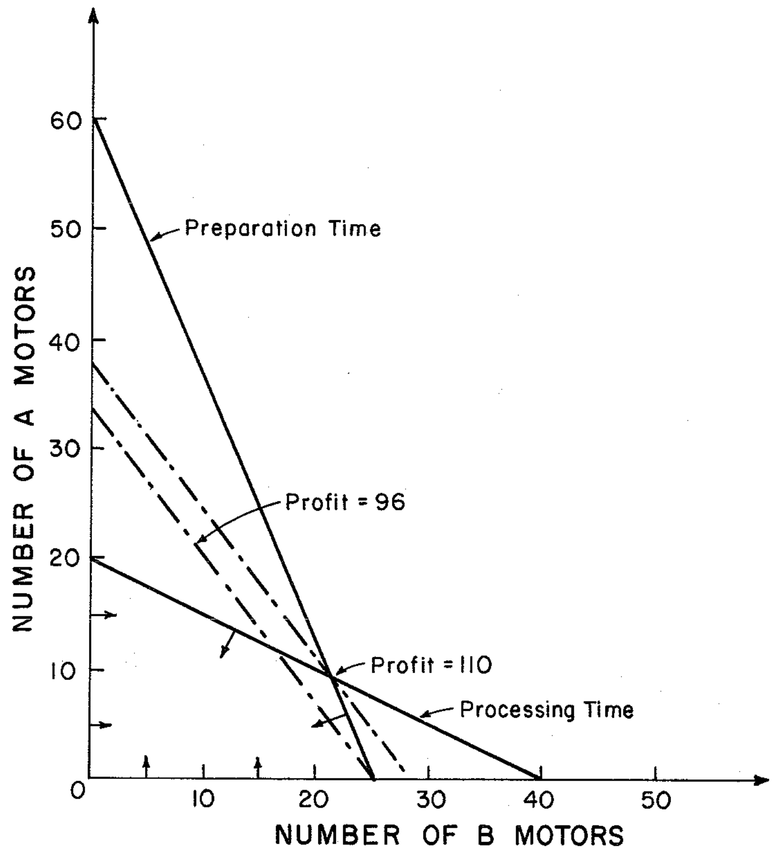

Figure 3-1 Constraints and Objective Function for Maximizing Rocket Motor Profit

Another geometric representation of the profit function and constraints is shown in Figure 3.2. The profit function is a plane and the highest point is the vertex A = *10*, *B* = 20. The intersection of the profit function and planes of *P* = constant give a line on the profit function plane as shown for *P* = 96. The projection of this line on the response surface (the *A* - *B* plane) is the same line shown in Figure 3.1 for *P* = 96. This diagram emphasizes the fact that the profit function is a plane, and the maximum profit will be at the highest point on the plane and located on the boundary at the intersection of constraint equations, a vertex.

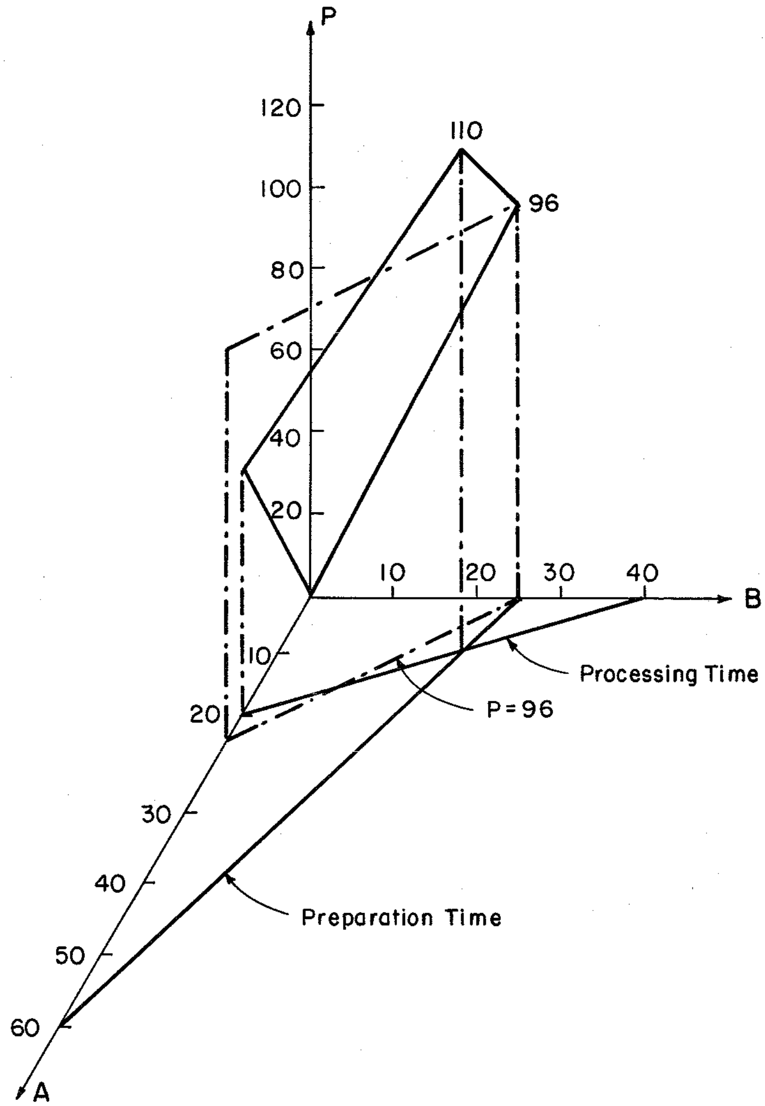

Figure 3-2 Geometric Representation of Constraints and Objective Function for Maximizing Rocket Motor Profit

This example can be used to illustrate *infeasibility* also, i.e., no feasible solution to linear programming problems. For example, if there were constraints on A and B such that $A \geq 21$ and $B \geq 25$, then there would be no solution since the processing and preparation time constraints could not be satisfied. Although it is obvious here that A and B could not have these values, it is not unusual in large problems to make a mistake and have the linear programming code return the result INFEASIBLE SOLUTION - the constraints are inconsistent. Almost always a blunder has been made, and the constraints do not represent the process. However, in large problems the blunder may not be obvious, and some effort may be required to find the error.

General Statement of the Linear Programming Problem

There are several ways to write the general mathematical statement of the linear programming problem. First, in the usual algebraic notation:

Objective Function:

$$\text{optimize:} \quad c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{3-1a}$$

Constraint Equations:

$$\text{subject to:} \quad a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$$

(4-1b)

$$
\begin{aligned}
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\geq b_2 \\
\vdots \qquad\qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\geq b_m
\end{aligned} \tag{3-1b}
$$

$$x_j \geq 0 \qquad \text{for } j = 1,2, \ldots n \tag{3-1c}$$

(4-1c)

We seek the values of the x_j's that optimize (maximize or minimize) the objective function, Equation (3-1a). The coefficients, c_j's, of the x_j's are referred to as *cost coefficients*. These can be positive and negative depending on the problem. Also, the values of the x_j's must satisfy the constraint equations, Equation (3-1b), and be nonnegative, Equation (3-1c).

There are more unknowns than constraint equations after the inequalities have been converted to equalities using slack variables. There will be m positive x_j's that optimize the objective function and the remaining $(n - m)$ x_j's will be zero. In a chemical or refinery process, the independent variables can be flow rates, for example; and the constraint equations can be material and energy balances, availability of raw materials, limits on process unit capacities, demands for products, etc.

The general formulation can also be written as:

$$\text{optimize: } \sum_{j=1}^{n} c_j x_j \qquad (4\text{-}2a) \qquad (3\text{-}2a)$$

$$\text{subject to: } \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \text{for } i = 1,2, \ldots m \qquad (4\text{-}2b) \qquad (3\text{-}2b)$$

$$x_j \geq 0 \qquad \text{for } j = 1,2, \ldots n \qquad (4\text{-}2c) \qquad (3\text{-}2c)$$

Matrix notation is another convenient method of writing the above equations.

$$\text{optimize: } \mathbf{c}^T \mathbf{x} \qquad (3\text{-}3a)$$

$$\text{subject to: } \mathbf{A\,x} \geq \mathbf{b} \qquad (3\text{-}3b)$$

$$\mathbf{x} \geq 0 \qquad (3\text{-}3c)$$

where

$$\mathbf{c}^T = [\, c_1, c_2, \ldots c_n \,]$$

$$\mathbf{x}^T = [\, x_1, x_2, \ldots x_n \,]$$

and

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The constraint equations given above have been written as inequalities. However, linear programming requires the constraints be equalities. In the next section, the use of slack and surplus variables is described to convert the inequalities to equalities.

Slack and Surplus Variables

In Example 3-1 the constraint equations were inequalities and the graphical method of locating the optimum was not affected by the constraints being inequalities. However, the computational method to determine the optimum, the Simplex Method, requires equality constraints. As was done in Chapter 2, the inequalities are converted to equalities by introducing slack and surplus variables. This is illustrated by converting the inequality, Equation 3-4, to an equality, Equation 3-5.

$$x_1 + x_2 \le b \tag{3-4}$$

Here a positive x_3 is being added to the left-hand side of Equation 3-4, and x_3 is the slack variable:

$$x_1 + x_2 + x_3 = b \tag{3-5}$$

If the inequality had been of greater than or equal to type, then a surplus variable would have been subtracted from the left-hand side of the equation to convert it to an equality.

In linear programming it is not necessary to use x_3^2, as in Chapter 2, since the computational method to find the optimum, the Simplex Method, does not allow variables to take on negative values. If the slack variable is zero, as it is in some cases, the largest value of the sum of the other variables ($x_1 + x_2$) is optimum, and the constraint is tight or active. If the slack variable is positive, then this would represent a difference or slack between the optimum values of ($x_1 + x_2$) and the total value that ($x_1 + x_2$) could have. In this case the constraint is loose or passive.

Basic and Basic Feasible Solutions of the Constraint Equations

Now let us focus on the constraint equation set alone, written as equalities (i.e., slack and surplus variables have been added), and discuss the possible solutions that can be obtained. This set can be written as:

$$\mathbf{A\,x} = \mathbf{b} \tag{3-6}$$

There are m equations and n unknowns where $n \ge m$ (for convenience using n again which now would include the slack and surplus variables, also).

A number of solutions can be generated for this set of linear algebraic equations by selecting (n - m) of the x_j's to be equal to zero. In fact, this number can be computed using the following formula (9).

$$\text{Maximum number of basic solutions} = \frac{n!}{m!(n-m)!} \tag{3-7}$$

Thus, a *basic solution* of the constraint equations is a solution obtained by setting (n - m) variables equal to zero and solving the constraint set for the remaining m variables. From this set of basic solutions, a group of solutions are selected where the values of the variables are all nonnegative, basic feasible solutions. The number of solutions can be estimated by the following formula (18).

$$\text{Approximate number of basic feasible solutions} = 2m \tag{3-8}$$

Thus, a *nondegenerate basic feasible solution* is a basic solution where all of the m variables are positive. A solution of m variables that are all positive is called a *basis* in the linear programming jargon.

Let us focus on the objective function, Equation 3-1a, now that we have a set of basic feasible solutions from the constraint equations. It turns out that one of the basic feasible solutions is the minimum of the objective function, and another one of these basic feasible solutions is the maximum of the objective function. The Simplex Algorithm begins at a basic feasible solution and moves to the maximum (or minimum) of the objective function stepping from one basic feasible solution to another with ever increasing (or decreasing) values of the objective function until the maximum (or minimum) is reached. The optimum is found in a finite number of steps, usually between *m* and 2*m* (7).

We will need to know how to obtain the first basic feasible solution and how to apply the Simplex Algorithm. Also, it will be seen that when the maximum (or minimum) is reached the algorithm has an automatic stopping procedure. Having briefly described the Simplex Method, let us give the procedure, illustrate its use with an example, and present some of the mathematical basis for the methodology in the next section.

Optimization with the Simplex Method

The Simplex Method is an algorithm that steps from one basic feasible solution (intersection of the constraint equations or vertex) to another basic feasible solution in a manner to have the objective function always increase or decrease. Without attempting to show a model associated with the following linear programming problem (2), let us see how the algorithm operates.

Example 3-2

For the following linear programming problem, convert then constraint equations to equality constraints using slack variables:

$$\text{maximize:} \quad x_1 + 2x_2$$

$$\text{subject to:} \quad 2x_1 + x_2 \le 10$$

$$x_1 + x_2 \le 6$$

$$-x_1 + x_2 \le 2$$

$$-2x_1 + x_2 \le 1$$

$$x_1 , \ x_2 \ge 0$$

When the slack variables are inserted, the constraint equations are converted to equalities, as shown below.

$$
\begin{aligned}
\text{Maximize:} \quad & x_1 + 2x_2 && = p \\
\text{Subject to:} \quad & 2x_1 + x_2 + x_3 && = 10 \\
& x_1 + x_2 + x_4 && = 6 \\
& -x_1 + x_2 + x_5 && = 2 \\
& -2x_1 + x_2 + x_6 && = 1 \\
& x_j \geq 0, \qquad j = 1, 2, ..., 6.
\end{aligned}
$$

where p represents the value of the objective function.

There are six variables in the set of four constraint equations in Example 3-2. To generate basic solutions, two of the variables are set equal to zero, and the equations are solved for the remaining four variables for the solution. This has been done (2), and all of the basic feasible solutions were selected from the basic solutions and listed in Table 3-1. These correspond to the vertices of the convex polygon *A-B-C-D-E-F* as shown in Figure 3-3. Also shown in Table 3-1 are the values of the objective function evaluated for each basic feasible solution. As can be seen, the maximum of the objective function is at the basic feasible solution, $x_1 = 2$, $x_2 = 4$ (Vertex D); and the minimum is at the basic feasible solution, $x_1 = 0$, $x_2 = 0$ (Vertex A).

Table 3-1. Basic Feasible Solutions of the Constraint Equations in Example 3-2

Vertex	x_1	x_2	x_3	x_4	x_5	x_6	p
A	0	0	10	6	2	1	0
B	0	1	9	5	1	0	2
C	1	3	5	2	0	0	7
D	2	4	2	0	0	1	10
E	4	2	0	0	4	7	8
F	5	0	0	1	7	11	5

The number of basic solutions is given by Equation 3-7 (5). For $n = 6$ and $m = 4$ the number of basic solutions is 15. One of the basic solutions of the constraint equations is obtained by setting $x_1 = x_4 = 0$, and the result is:

$$x_1 = 0, \quad x_2 = 6, \quad x_3 = 4, \quad x_4 = 0, \quad x_5 = -4 \quad \text{and} \quad x_6 = -5$$

Here two of the four values of the variables are negative. The approximate number of basic feasible solutions given by Equation (3-6) is eight, which is close to the actual number of six.

Referring to Table 3-1 and Figure 3-3 and comparing the variables in a basis with those in an adjacent basis, it is seen that each have all but one nonzero variable in common. For example, to obtain basis *B* from basis *A* it is necessary to remove x_6 from the basis (i.e., set x_6 = zero) and bring x_2 into the basis (i.e., solve for $x_2 \neq 0$). The Simplex Method does this and moves

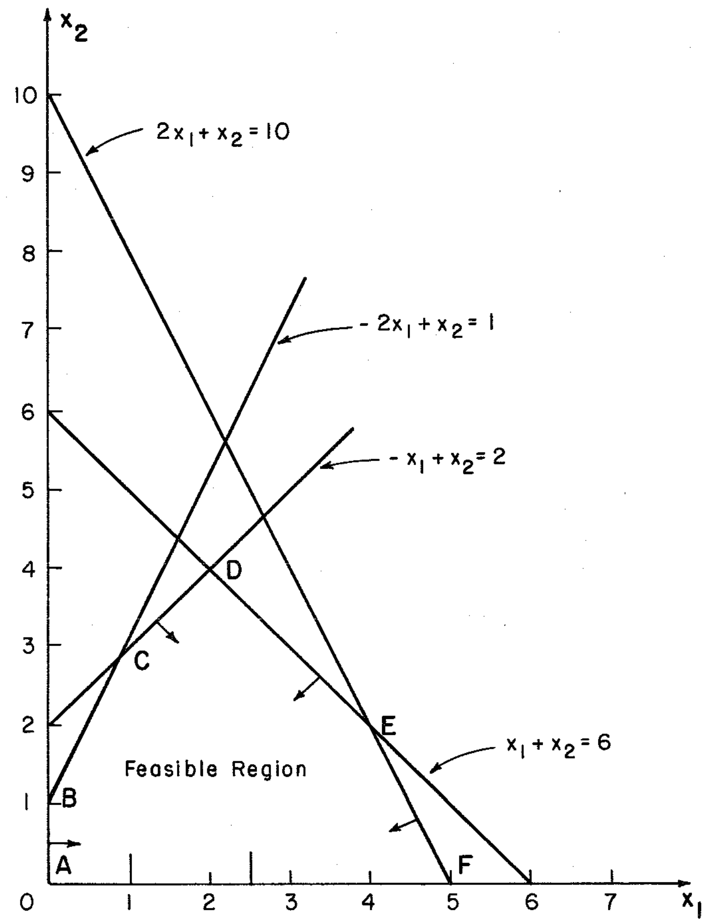

Figure 3-3 Geometric Representation of the Constraints in Example 4-2

from one basic feasible solution to another. Each time it moves in a direction of an improved value of the objective function. This is the key to the Simplex Algorithm. To move in this fashion only requires the use of Gaussian elimination applied to the constraints and then to the objective function to determine its new improved value.

The procedure to solve a linear programming problem using the *Simplex Algorithm to maximize the objective function* is:

1. Place the problem in a linear programming format with linear constraint equations and linear objective function.

2. Introduce slack and surplus variables to convert inequalities to equalities and adjust the constraint equations to have positive right-hand sides.

3. Select an initial basic feasible solution. If all of the constraint equations were inequalities of the less than or equal to form, the slack variables can be used as the initially feasible basis.

4. Perform algebraic manipulations to express the objective function in terms of variables that are not in the basis, i.e., are equal to zero. This determines the value of the objective function for the variables in the basis.

5. Inspect the objective function and select the variable with the largest positive coefficient to bring into the basis, i.e., make nonzero. If there are no positive coefficients, the maximum has been reached (automatic stopping feature of the algorithm).

6. Inspect the constraint equations to select the one to be used for algebraic manipulations to change the variable in the basis. The selection is made to have positive right-hand sides from the Gaussian elimination. This is necessary to guarantee that all of the variables in the new basis will be positive. Use this equation to eliminate the variable selected in step 5 from all of the other constraint equations.

7. Use the constraint equation selected in step 6 to eliminate the variable selected in step 5 from the objective function. This moves one of the variables previously in the basis to the objective function, and it is dropped from the basis, i.e. set equal to zero. Also, this determines the new value of the objective function.

8. Repeat the procedure of steps 5 through 7 until all coefficients in the objective function are negative and stop. If the procedure is continued past this point, then the value of the objective function would decrease. This is the automatic stopping feature of the algorithm.

The Simplex Algorithm will be applied to Example 3-2 to illustrate the computational procedure. The first two steps have been completed, and the slack variables will be used as the initial feasible basis (Step 3).

Example 3-3

Apply the Simplex Method to the linear programming problem of Example 3-2 using the slack variables as the first basic feasible solution.

maximize: $x_1 + 2x_2 = p \qquad p = 0$

subject to: $2x_1 + x_2 + x_3 = 10 \qquad x_3 = 10$

$x_1 + x_2 + x_4 = 6 \qquad x_4 = 6$

$-x_1 + x_2 + x_5 = 2 \qquad x_5 = 2$

$$-2x_1 + x_2 \qquad + x_6 = 1 \qquad x_6 = 1$$

$$x_1 = 0$$
$$x_2 = 0$$

Continuing with the procedure, x_2 is the variable in the objective function with the largest positive coefficient. Thus, increasing x_2 will increase the objective function (step 5).

The fourth constraint equation will be used to eliminate x_2 from the objective function (step 6). The variable x_2 is said to enter the basis, and x_6 is to leave.

Proceeding with the Gaussian elimination gives:

maximize:	$5x_1 \qquad -2x_6 = p - 2$	$p = 2$
subject to:	$4x_1 + x_3 - x_6 = 9$	$x_3 = 9$
	$3x_1 + x_4 - x_6 = 5$	$x_4 = 5$
	$x_1 + x_5 - x_6 = 1$	$x_5 = 1$
	$-2x_1 + x_2 + x_6 = 1$	$x_2 = 1$
		$x_1 = 0$
		$x_6 = 0$

The nonzero variables in the basis are x_2, x_3, x_4, and x_5; and the objective function has increased from $p = 0$ to $p = 2$.

The procedure is repeated (Step 8) selecting x_1 to enter the basis. The third constraint equation is used, and x_5 leaves the basis. Performing the manipulations gives:

maximize:	$-5x_5 + 3x_6 = p - 7$	$p = 7$
subject to:	$x_3 - 4x_5 + 3x_6 = 5$	$x_3 = 5$
	$x_4 - 3x_5 + 2x_6 = 2$	$x_4 = 2$
	$x_1 + x_5 - x_6 = 1$	$x_1 = 1$
	$x_2 + 2x_5 - x_6 = 3$	$x_2 = 3$
		$x_5 = 0$
		$x_6 = 0$

The procedure is repeated, and x_6 is selected to enter the basis. The second constraint equation is used, and x_4 leaves the basis. The results of the manipulations are:

maximize:	$- 3/2\, x_4 - 1/2\, x_5$	$= p - 10$	$p = 10$
subject to:	$x_3 - 3/2\, x_4 + 1/2\, x_5$	$= 2$	$x_3 = 2$
	$1/2\, x_4 - 3/2\, x_5 + x_6$	$= 1$	$x_6 = 1$
	$x_1 + 1/2\, x_4 + 1/2\, x_5$	$= 2$	$x_1 = 2$
	$x_2 + 1/2\, x_4 + 1/2\, x_5$	$= 4$	$x_2 = 4$
			$x_4 = 0$
			$x_5 = 0$

All of the coefficients in the objective function are negative for the variables that are not in the basis. If x_4 or x_5 were increased from zero to a positive value, the objective function would decrease. Thus, the maximum is reached, and the optimal basic feasible solution has been obtained.

Referring to Table 3-1 and Figure 3-3 for the set of basic feasible solutions, it is seen that the Simplex Method started at vertex *A*. The first application of the procedure stepped to the adjacent vertex *B*, with an increase in the objective function to 2. Proceeding, the Simplex Method then moved to vertex *C*, where the objective function increased to 7. At the next application of the algorithm, the optimum was reached at vertex *D* with $p = 10$. At this point the application of the Simplex Method stopped since the maximum had been reached.

Let us use this example to demonstrate that the Simplex Method can be used to find the minimum of an objective function by only slightly modifying the logic of the algorithm for maximizing the objective function. If we begin by minimizing the objective function given in the last step of Example 3-3, the largest decrease in the objective function is made by selecting x_4 to enter the basis (Step 5), i.e., selecting the variable which is not in the basis and whose coefficient is the largest in absolute value and negative. Then select the second constraint equation for the manipulations to have positive right-hand sides of the constraints. This has x_4 entering the basis, and x_6 leaving the basis. The results are the same as in the next to last step of the example. Proceeding, x_5 is selected to enter the basis, the third constraint equation is used for the manipulations, and x_1 leaves the basis. The results are the same as the second step of the example. Continuing, x_6 is selected to enter the basis, the fourth constraint equation is used for the manipulations, and x_2 leaves the basis. The results are the same as the first step in the example, and all of the coefficients of the variables in the objective function are positive for the variables not in the basis. The minimum has been reached, because if either x_1 or x_2 were brought into the basis, i.e., made positive, the objective function would increase.

Thus, the Simplex Algorithm applies for either maximizing or minimizing the objective function. The logic of the algorithm is essentially the same in both cases, and it only differs in the selection of the variable to enter the basis, i.e., largest positive coefficient for maximizing or the largest in absolute value and negative for minimizing.

With this example we have illustrated the computational procedure of the Simplex Algorithm. Also, we have seen that a solution of the constraints gives the maximum of the objective function, and another solution gives the minimum of the objective function. These results can be proven mathematically to be true for the linear programming problem stated as Equation 3-1, and the details are given in texts devoted to linear programming. In the following section we will give a standard tabular method for the Simplex Method, and then the key theorems of linear programming will be presented along with a list of references where more details can be found on mathematical aspects of linear programming.

Simplex Tableau

In using the Simplex Method, it is not necessary to write the x_j symbols when doing the Gaussian elimination procedure, and a standard method for hand computations has been developed which uses only the coefficients of the objective function and constraints in a series of tables. This is called the Simplex Tableau, and this procedure will be illustrated using the problem given in Example 3-3.

The Simplex Tableau for the three applications of the Simplex Algorithm of Example 3-3 is shown in Figure 3-4. In this table, dots have been used in places that have to be zero, as opposed to just turning out to be zero. Also, the objective function has been set equal to $-y$, because the tableau procedure minimizes the objective function and is called z, i.e., $z = -y = -x_1 - 2x_2$. Then the objective function is included in the last row of the tableau as $-z - x_1 - 2x_2 = 0$ to have the same form as the constraint equations. Iteration 0 in Table 3-4 is the initial tableau.

The slack variables are the initially feasible basis in this example, and the Simplex Algorithm first locates the smallest coefficient in the objective function of the variables not in the basis. In this case it is x_2 as shown in Figure 3-4 with a coefficient of -2; x_2 will enter the basis, i.e., becomes positive. A pivotal element is located to insure the next basis is feasible using a minimum ratio test, i.e., selecting the smallest value of (10/1, 6/1, 2/1, 1/1), and the pivotal element is indicated as an asterisk identifying the pivotal row used for the Gaussian elimination to move to iteration 1, with x_6 leaving the basis.

The above procedure is repeated for two more iterations, as shown in Figure 3-4. The pivotal elements are indicated by an asterisk, having been located by the minimum ratio test. The procedure ends when the values in the objective function row are all positive, for this is a minimizing problem. Also, a comparison of the results in Figure 3-4 with those in Example 3-3 shows the concise nature of the Simplex Tableau. In addition, if a pivotal element cannot be located using the minimum ratio test, this means that the problem has an unbounded solution, or a blunder has been made.

Iteration	Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6
0	x_3	10	2	1	1	·	·	·
	x_4	6	1	1	·	1	·	·
	x_5	2	-1	1	·	·	1	·
	x_6	1	-2	1*	·	·	·	1
	$-z$	0	-1	-2	·	·	·	·

Initial tableau, x_2 enters basis, x_6 leaves the basis.

Iteration	Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6
1	x_3	9	4	·	1	·	·	-1
	x_4	5	3	·	·	1	·	-1
	x_5	1	1*	·	·	·	1	-1
	x_2	1	-2	1	·	·	·	1
	$-z$	2	-5	·	·	·	·	2

First iteration, x_1 enters the basis, x_5 leaves the basis.

Iteration	Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6
2	x_3	5	·	·	1	·	-4	3
	x_4	2	·	·	·	1	-3	2*
	x_1	1	1	·	·	·	1	-1
	x_2	3	·	1	·	·	2	-1
	$-z$	7	·	·	·	·	5	-3

Second iteration, x_6 enters the basis, x_4 leaves the basis.

Iteration	Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6
3	x_3	2	·	·	1	-3/2	1/2	·
	x_6	1	·	·	·	1/2	-3/2	1
	x_1	2	1	·	·	1/2	-1/2	·
	x_2	4	·	1	·	1/2	1/2	·
	$-z$	10	·	·	·	3/2	1/2	·

Final iteration, coefficients are positive, minimum has been reached

*Pivotal element from the minimum ratio test

Figure 3-4 Illustration of the Simplex Tableau

The Simplex Tableau procedure can be used effectively for hand calculations when artificial variables are employed to start the solution with an initially feasible basis and to identify problems such as degeneracy. The topics of degeneracy and artificial variables will follow the discussion of the mathematics of linear programming.

Mathematics of Linear Programming

The mathematics of convex sets and linear inequalities has to be developed to prove the theorems that establish the previous procedure for locating the optimal solution of the linear programming problem. This theory is done in many of the standard texts devoted to the subject and is beyond the scope of this brief discussion. However, the appropriate theorems will be given with an explanation, to convey these concepts. Those who are interested in further details are referred to standard works such as Garvin (3) or Gass (7).

A *feasible solution*, is any solution to the constraint equations, Equation 3-1 and also, is a convex set. A *convex set* is illustrated in Figure 3-5a, for two dimensions and is a collection of points such that if it contains any two points *A* and *B*, is also contains the straight-line *AB* between the points. An example of a nonconvex set is shown in Figure 3-5b. Also, an *extreme point* or *vertex* of a convex set is a point that does not lie on any segment joining two other points in the set.

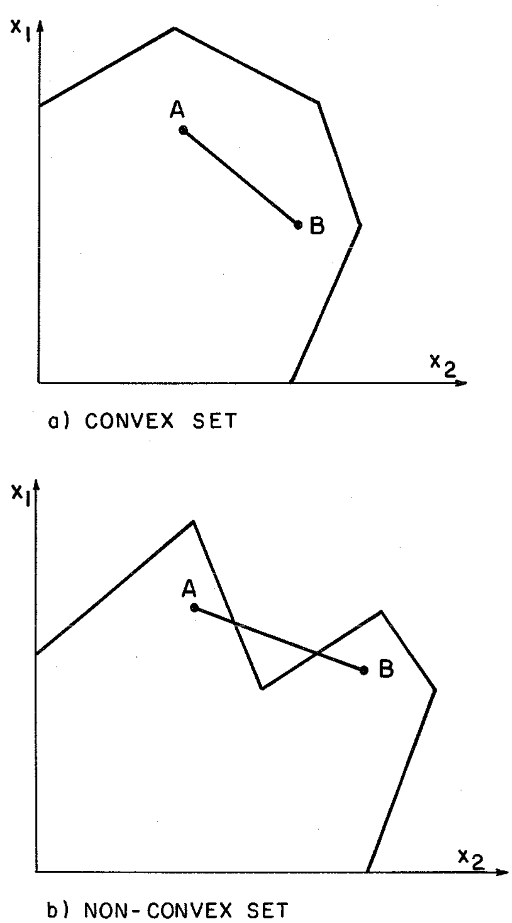

Figure 3-5 Convex and Nonconvex Sets

The important theorem relating convex sets with feasible and basic feasible solutions is:

The collection of feasible solutions constitutes a convex set whose extreme points correspond to basic feasible solutions. (4)

In the proof of the above theorem it is shown that a linear combination of any two feasible solutions is a feasible solution and hence lies on a straight line between the two. Thus, this constitutes a convex set. To prove that a basic feasible solution is an extreme point, it is assumed that a basic feasible solution can be expressed as a linear combination of feasible solutions. Then it is shown by contradiction that this is impossible. Thus, it must be an extreme point.

The next important theorem is an existence theorem:

If a feasible solution exists, then a basic feasible solution exists. (5)

This theorem is proved by showing that a basic feasible solution can be constructed from a feasible solution.

The next theorem relates the maximum or minimum of the objective function to the basic feasible solutions of the constraint equations.

If the objective function possesses a finite minimum, then at least one optimal solution is a basic feasible solution. (6)

This theorem can be proved by writing a solution to the constraint equations as the weighted sum of a feasible solution and a basic solution where a range on the weights determines that this solution of the constraint equations is a feasible solution. The objective function can then be put in the form of the weights, and limits on the weights are determined that has the feasible solution be a basic feasible solution. Next, it can be shown that it is always possible to generate a new feasible solution which contains at least one more variable at zero than the current one, and the new value of the objective function will be less than or equal to the current value. Continuing to generate new feasible solutions by the procedure has the feasible solutions become a basic feasible solution, if the objective function is not equal to minus infinity. The procedure holds for any feasible solution, and then it holds for an optimal solution. Thus, the optimal solution is a basic feasible solution. Details of this proof are given in Garvin (6).

This theorem provides the basis for locating optimal solutions of the linear programming problem. Only basic feasible solutions need to be examined to determine the maximum and minimum for the problem, and there are a finite number of basic feasible solutions. In contrast there are an infinite number of feasible solutions.

To formalize the simplex computational procedure, consider the set of equations with a basic feasible solution $\mathbf{x} = (x_4, x_5, x_6)$.

$$\text{maximize:} \quad c_1x_1 + c_2x_2 + c_3x_3 = p_0 \qquad (3\text{-}9a)$$

$$\begin{aligned} \text{subject to:} \quad & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + x_4 = b_1 \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + x_5 = b_2 \qquad (3\text{-}9b) \\ & a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + x_6 = b_3 \\ & x_j \geq 0 \qquad i = 1,2, ...6 \end{aligned}$$

If c_1 is the largest positive coefficient and b_1/a_{11} is the smallest positive ratio, then x_1 enters the basis and x_4 leaves the basis. Performing the elimination, the result is:

$$\text{maximize:} \quad (c_2 - c_1a_{12}/a_{11})x_2 + (c_3 - c_1a_{13}/a_{11})x_3 - (c_1a_{14}/a_{11})x_4 = p_o - c_1b_1/a_{11} = p_1 \qquad (3\text{-}10a)$$

$$\begin{aligned} \text{subject to:} \quad & x_1 + (a_{12}/a_{11})x_2 + (a_{13}/a_{11})x_3 + (a_{14}/a_{11})x_4 = b_1a_{11} \qquad (3\text{-}10b) \\ & (a_{22}-a_{21}a_{12}/a_{11})x_2 + (a_{23}-a_{21}a_{13}/a_{11})x_3 - (a_{21}a_{14}/a_{11})x_4 + x_5 = b_2-a_{21}b_1/a_{11} \\ & (a_{32}-a_{31}a_{12}/a_{11})x_2 + (a_{33}-a_{31}a_{13}/a_{11})x_3 - (a_{31}a_{14}/a_{11})x_4 + x_6 = b_3-a_{31}b_1/a_{11} \end{aligned}$$

If $p_1 > p_o$, then there is an improvement in the objective function, and the solution is continued. If $p_1 < p_o$, then no improvement in the objective function is obtained, and **x** is the basic feasible solution that maximizes the objective function. The following theorem given by Gass (7) is:

> *If for any basic feasible solution $x_k = (x_1, x_2, ... x_m)$ the condition $p(\mathbf{x}_k) > p(\mathbf{x}_j)$ for all $j = 1,2,...n$ $(j \neq k)$ hold, then x_k is a basic feasible solution that maximized the objective function.*

The proof of this theorem is similar to that of the previous theorem. Also, a corresponding result can be obtained for the basic feasible solution that minimizes the objective function.

Further information is given in the textbooks by Garvin (6), Gass (7), and others listed in the table on selected texts given at the end of the chapter. These books give detailed proofs to the key theorems and other related ones.

Degeneracy

In the Simplex Method there is an improvement in the objective function in each step as the algorithm converges to the optimum. However, a situation can arise where there is no improvement in the objective function from an application of the algorithm, and this is referred to as *degeneracy*. Also, there is a possibility that cycling could occur, and the optimum would not be reached. Degeneracy occurs when the right-hand side of one of the constraint equations is equal to zero, and this equation is selected for the algebraic manipulation to change variables in the basis and evaluate the objective function. Graphically this occurs when two vertices coalesce into one vertex. It is reported (6) that it is not unusual for degeneracy to occur in the various applications of linear programming. However, there has not been a case of cycling reported. An example of cycling has been constructed, and a procedure to prevent cycling has been developed. However, these are not usually employed. The following example from Garvin (6) illustrates degeneracy, and an optimal solution is found even if it does occur.

Example 3-4

Solve the following problem by the Simplex Method.

maximize: $2x_1 + x_2$

subject to:

$$x_1 + 2x_2 \leq 10$$

$$x_1 + x_2 \leq 6$$

$$x_1 - x_2 \leq 2$$

$$x_1 - 2x_2 \leq 1$$

$$2x_1 - 3x_2 \leq 3$$

A graphical representation of the constraint equations is shown in Figure 3-6. It shows that the last three constraint equations all intersect at vertex *C*. Vertex *C* is said to be overdetermined. If the constraint equation $x_1 - 2x_2 \leq 1$ had been $0.9x_1 - 2x_2 \leq 1$, there would have been two separate vertices, as shown in Figure 3-6. Degeneracy occurs when two or more vertices coalesce into a single vertex.

To illustrate what happens, the Simplex Algorithm will be started at vertex *A* and move through *B* and *C* to *D*, where the optimal solution is $p = 10$ for $x_1 = 4$ and $x_2 = 2$. Using the slack variables as the initially feasible basis gives:

		Vertex A
$2x_1 + x_2$	$= p$	$p = 0$
$x_1 + 2x_2 + x_3$	$= 10$	$x_3 = 10$
$x_1 + x_2 + x_4$	$= 6$	$x_4 = 6$
$x_1 - x_2 + x_5$	$= 2$	$x_5 = 2$
⇨ $x_1 - 2x_2 + x_6$	$= 1$	$x_6 = 1$
$2x_1 - 3x_2 + x_7$	$= 3$	$x_7 = 3$

Then x_1 is selected to enter the basis and x_6 leaves the basis. Performing the algebraic manipulations, the following results are obtained for vertex *B*.

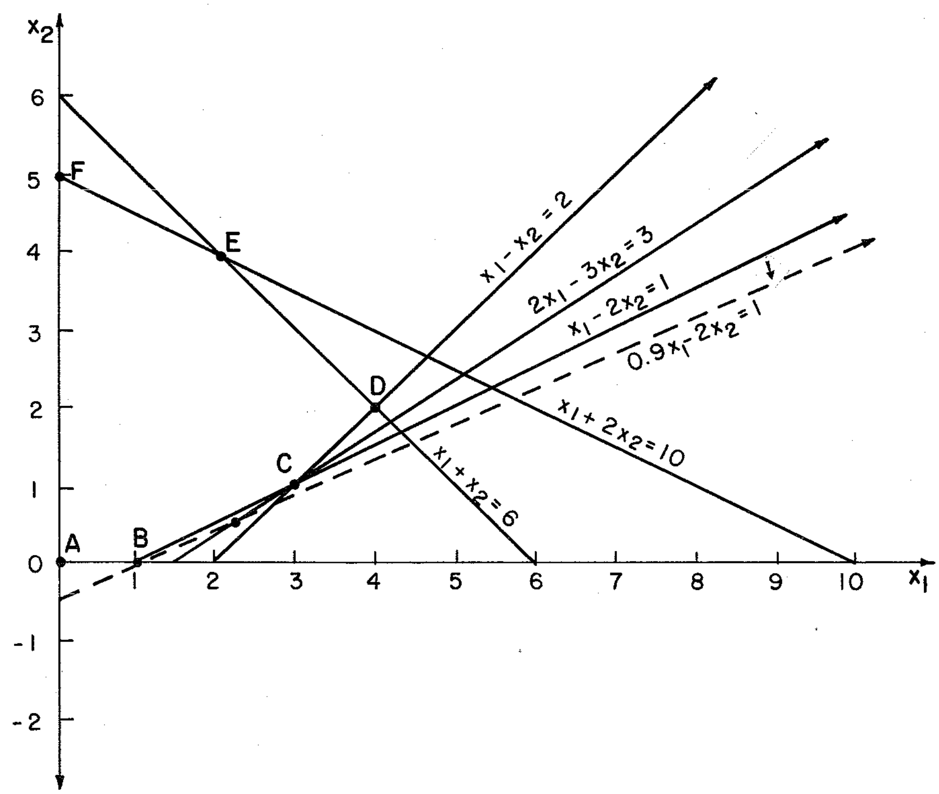

Figure 3-6 Graphical Representation of the Constraint Equations for Example 4-4

Vertex *B*

	$5x_2 \qquad\qquad\qquad - 2x_6$	$= p - 2$	$p = 2$
	$4x_2 + x_3 \qquad\qquad - x_6$	$= 9$	$x_3 = 9$
	$3x_2 \qquad + x_4 \qquad - x_6$	$= 5$	$x_4 = 5$
⇨	$x_2 \qquad\qquad + x_5 \quad - x_6$	$= 1$	$x_5 = 1$
	$x_1 - 2x_2 \qquad\qquad\quad + x_6$	$= 1$	$x_1 = 1$
⇨	$x_2 \qquad\qquad\qquad - 2x_6 + x_7$	$= 1$	$x_7 = 1$

Then x_2 is selected to enter the basis and either the equation with x_5 or the equation with x_7 can be used for the algebraic manipulations. The following calculations use the equation with x_7 and then use the one with x_5 to illustrate the effect of these decisions. (In a computer program the decision would be made rather arbitrarily, e.g., by selecting the one with the lowest subscript.)

Performing the algebraic manipulations to have x_7 leave the basis gives:

		Vertex C
	$+ 8x_6 - 5x_7 = p - 7$	$p = 7$
	$x_3 + 7x_6 - 4x_7 = 5$	$x_3 = 5$
	$x_4 + 5x_6 - 3x_7 = 2$	$x_4 = 2$
$\Rightarrow$	$x_5 + x_6 - x_7 = 0$	$x_5 = 0$
	$x_1 - 3x_6 + 2x_7 = 3$	$x_1 = 3$
	$x_2 - 2x_6 + x_7 = 1$	$x_2 = 1$

The right-hand side of the third constraint equation is zero, and this causes $x_5 = 0$ which contradicts the fact that variables in the basis are to be greater than zero.

However, the procedure is to continue with the Simplex Method selecting x_6 to enter the basis, and the third constraint equation is used for the algebraic manipulations to have positive (or zero) right-hand sides. Then x_5 leaves the basis, and the result is:

		Vertex C
	$- 8x_5 + 3x_7 = p - 7$	$p = 7$
	$x_3 - 7x_5 + 3x_7 = 5$	$x_3 = 5$
$\Rightarrow$	$x_4 - 5x_5 + 2x_7 = 2$	$x_4 = 2$
	$x_5 + x_6 - x_7 = 0$	$x_6 = 0$
	$x_1 + 3x_5 - x_7 = 3$	$x_1 = 3$
	$x_2 + 2x_5 - x_7 = 1$	$x_2 = 1$

There was no improvement in the objective function and the Simplex Method did not move from vertex *C*.

The procedure is continued having x_7 enter the basis and x_4 leave the basis. The results of the algebraic manipulations are:

$$
\begin{array}{llllllllr}
 & & & & & & & & \text{Vertex } D \\
 & & -3/2\,x_4 & -\tfrac{1}{2}\,x_5 & & & = p - 10 & & p = 10 \\
 & x_3 & -3/2\,x_4 & +\tfrac{1}{2}\,x_5 & & & = 2 & & x_3 = 2 \\
 & & \tfrac{1}{2}\,x_4 & -5/2\,x_5 & & +x_7 & = 1 & & x_7 = 1 \\
 & & \tfrac{1}{2}\,x_4 & -3/2\,x_5 & +x_6 & & = 1 & & x_6 = 1 \\
x_1 & & +\tfrac{1}{2}x_4 & +\tfrac{1}{2}x_5 & & & = 4 & & x_1 = 4 \\
x_2 & & +\tfrac{1}{2}\,x_4 & -\tfrac{1}{2}\,x_5 & & & = 2 & & x_2 = 2
\end{array}
$$

The maximum has been reached since the coefficients of the variables in the objective function are all negative. The simplex algorithm was unaffected by the right-hand side of one of the equations becoming zero during the application of the algorithm.

Now returning to vertex B and selecting x_5 to enter the basis, the result of the manipulations is:

$$
\begin{array}{llllllllr}
 & & & & & & & & \text{Vertex } C \\
 & & & -5x_5 & +3x_6 & & = p - 7 & & p = 7 \\
 & x_3 & & -4x_5 & +3x_6 & & = 5 & & x_3 = 5 \\
 & & \Rightarrow\ x_4 & -3x_5 & +2x_6 & & = 2 & & x_4 = 2 \\
x_2 & & & +x_5 & -x_6 & & = 1 & & x_2 = 1 \\
x_1 & & & +2x_5 & -x_6 & & = 3 & & x_1 = 3 \\
 & & & -x_5 & -x_6 & +x_7 & = 0 & & x_7 = 0
\end{array}
$$

Then selecting x_6 to enter the basis and x_4 to leave the basis, the result of the manipulation is the optimum given at vertex D previously. Consequently, when using x_5 there is an improvement in the objective function and one fewer applications of the Simplex Algorithm were required.

Unfortunately, the effect of a constraint equation selection with degeneracy cannot be predicted in advance for large problems, and an arbitrary selection is made, as previously mentioned. In conclusion, degeneracy is not unusual, but it has yet to affect the solution of linear programming problems in industrial applications.

Artificial Variables

To start a linear programming problem, it is necessary to have an initially feasible basis as required in Step 3 of the Simplex Method and as shown in Equation (3-9b). In the illustrations up to now we have been able to use the slack variables as the initially feasible basis. However, the constraints generally are not in such a convenient form, so another procedure is used to have an initially feasible basis, artificial variables. In this technique a new variable, an artificial variable, is added to each constraint equation to give an initial feasible basis to start the solution. This is permissible, and it can be shown that the optimal solution to the original problem is the optimal solution to the problem with artificial variables. However, it is necessary to modify the objective function to ensure that all of the artificial variables leave the basis. This is accomplished by adding terms to the objective functions that consist of the product of each artificial variable and a negative coefficient that can be made arbitrarily large in magnitude for the case of maximizing the objective function. Thus, this will insure that the artificial variables are the first ones to leave the basis during the application of the Simplex Method.

At this point it is reasonable to question if this would not be a significant amount of computations for convenience only. The answer would be yes if only one small linear programming problem was to be solved. However, this is not usually the case, and the margin for error is reduced significantly by avoiding manipulation of the constraint equations in a large problem. In fact, large linear programming codes only require the specification of the values of the coefficients in the objective function and the coefficients, right-hand sides and the types of inequalities of the constraint equations to obtain an optimal solution. These programs can solve linear programming problems having thousands of constraints and thousands of variables (12). Consequently, developing a linear model of a plant or a process is the main effort required, and then one of the available general linear programming codes can be used to obtain the optimal solution. Also, most major companies have a group that includes experts in using linear programming; and also, there are firms that specialize in industrial applications of linear programming.

The following example illustrates the use of artificial variables as they might be employed in a computer program. The technique is sometimes called the "big *M* method." Another method, the "Two-Phase" method is comparable. See Problem 3-25.

Example 3-5 (8)

Solve the following linear programming problem using artificial variables.

$$\text{minimize:} \quad x_1 + 3x_2$$

$$\text{subject to:} \quad x_1 + 4x_2 \geq 24$$

$$5x_1 + x_2 \geq 25$$

Slack variables x_3 and x_4 and artificial variables a_1 and a_2 are introduced as shown below. The artificial variables will be the initially feasible basis since the slack variable would give negative values, and algebraic manipulations would be required to have x_1 and x_2 be the initially feasible basis. In the objective function M is the coefficient of the artificial variables a_1 and a_2, and M can be made arbitrarily large to drive a_1 and a_2 from the basis.

$$\text{minimize:} \quad x_1 + 3x_2 + M a_1 + M a_2 = c$$

$$\text{subject to:} \quad x_1 + 4x_2 - x_3 + a_1 = 24$$

$$5x_1 + x_2 - x_4 + a_2 = 25$$

The two constraints equations are used to eliminate a_1 and a_2 from the objective function. This is Step 4 in the Simplex Method, and the objective function is a large number, $49M$, as shown below.

$$(1 - 6M)\, x_1 + (3 - 5M)\, x_2 + M x_3 + M x_4 = c - 49M \qquad c = 49M$$

$$x_1 + 4x_2 - x_3 + a_1 = 24 \qquad a_1 = 24$$

$$5x_1 + x_2 - x_4 + a_2 = 25 \qquad a_2 = 25$$

Applying the Simplex Algorithm, x_1 enters the basis since it has the negative coefficient that is largest in magnitude. The second constraint equation is used to perform the algebraic manipulations, and a_2 leaves the basis. Performing the manipulations gives:

$$(14/5 - 19/5M)\, x_2 + Mx_3 + (1/5 - 1/5M)\, x_4 - (1/5 - 6/5M)\, a_2 = c - 19M - 5$$

$$19/5x_2 - x_3 + 1/5x_4 + a_1 - 1/5a_2 = 19$$

$$x_1 + 1/5x_2 - 1/5x_4 + 1/5a_2 = 5$$

$$c = 19M + 5 \qquad a_1 = 19 \qquad x_1 = 5$$

Continuing with the Simplex Algorithm x_2 enters the basis. The first constraint equation is used for the algebraic manipulations, and a_1 leaves the basis. Performing the manipulations gives:

$$14/19\ x_3 + 5/95\, x_4 + (-14/19 + M)\, a_1 + (-5/95 + M)a_2 = c - 19$$

$$x_2 - 5/19x_3 + 1/19\, x_4 + 5/19\, a_1 - 1/19a_2 = 5$$

$$x_1 + 1/19x_3 - 20/95\, x_4 - 1/19a_1 + 20/95a_2 = 4$$

$$c = 19 \qquad x_1 = 4 \qquad x_2 = 5$$

Now the terms containing the artificial variables a_1 and a_2 can be dropped from the objective function and the constraint equations. The reason is that they both have large positive coefficients in the objective function and will not reenter the basis. The problem is continued without them to reduce computational effort. However, for this problem the optimum has been reached since all of the coefficients in the objective function are positive, and no further reduction can be obtained.

In addition to the infeasible difficulty, there is another problem that can be encountered in linear programming, an *unbounded problem*, which is usually caused by a blunder. In this situation, the constraint equations do not confine the variables to finite values. This is illustrated by changing the linear programming problem in Example 3-5 from one of minimizing $x_1 + 3x_2$ to maximizing $x_1 + 3x_2$ subject to the constraints given in the problem. The constraints are of the greater than or equal to type, and they are satisfied with values of $x_1 \geq 4$ and $x_2 \geq 5$. Then for maximizing the objective function the values of x_1 and x_2 could be increased without bounds to have the objective function also increase without bounds. Thus, the problem is said to be unbounded.

Formulating the Linear Programming Problem – A Simple Refinery

To this point in the discussion of linear programming the emphasis has been on the solution of problems by the Simplex Method. In this section procedures will be presented for the formulation of the linear programming problem for a plant or process. This will include developing the objective function from the cost or profit of the process or plant and the constraint equations from the availability of raw materials, the demand for products and equipment capacity limitations and conversion capabilities. A simple petroleum refinery will be used as an example to illustrate these procedures. Also, an optimal solution will be obtained using a large linear programming code to illustrate the use of one of these types of programs available on a large computer. In the following section the optimal solution of the general linear programming problem will be extended to a sensitivity analysis, and these results will be illustrated using the information computed from the large linear programming code for the simple refinery example.

In Figure 3-7 the flow diagram for the simple petroleum refinery is shown, and in Table 3-2 the definition is given for the name of each of the process streams. There are only three process units in this refinery, and these are a crude oil atmospheric distillation column, a catalytic cracking unit and a catalytic reformer. The crude oil distillation column separates crude oil into five streams which are fuel gas, straight run gasoline, straight run naphtha, straight run distillate and straight run fuel oil. Part of the straight run naphtha is processed through the catalytic reformer to improve its quality, i.e., increase the octane number. Also, parts of the straight run distillate and straight run fuel oil are processed through the catalytic cracking unit to improve their quality so they can be blended into gasoline. The refinery produces four products, and these are premium gasoline, regular gasoline, diesel fuel and fuel oil. Even for this simple refinery there are 33 flow rates for which the optimal values have to be determined. This small problem points out one of the difficulties of large linear programming problems. The formulation of the problem is quite straightforward. However, there is a major accounting problem in keeping track of a large number of variables, and the collection of reliable data to go with these variables is usually very time consuming (9).

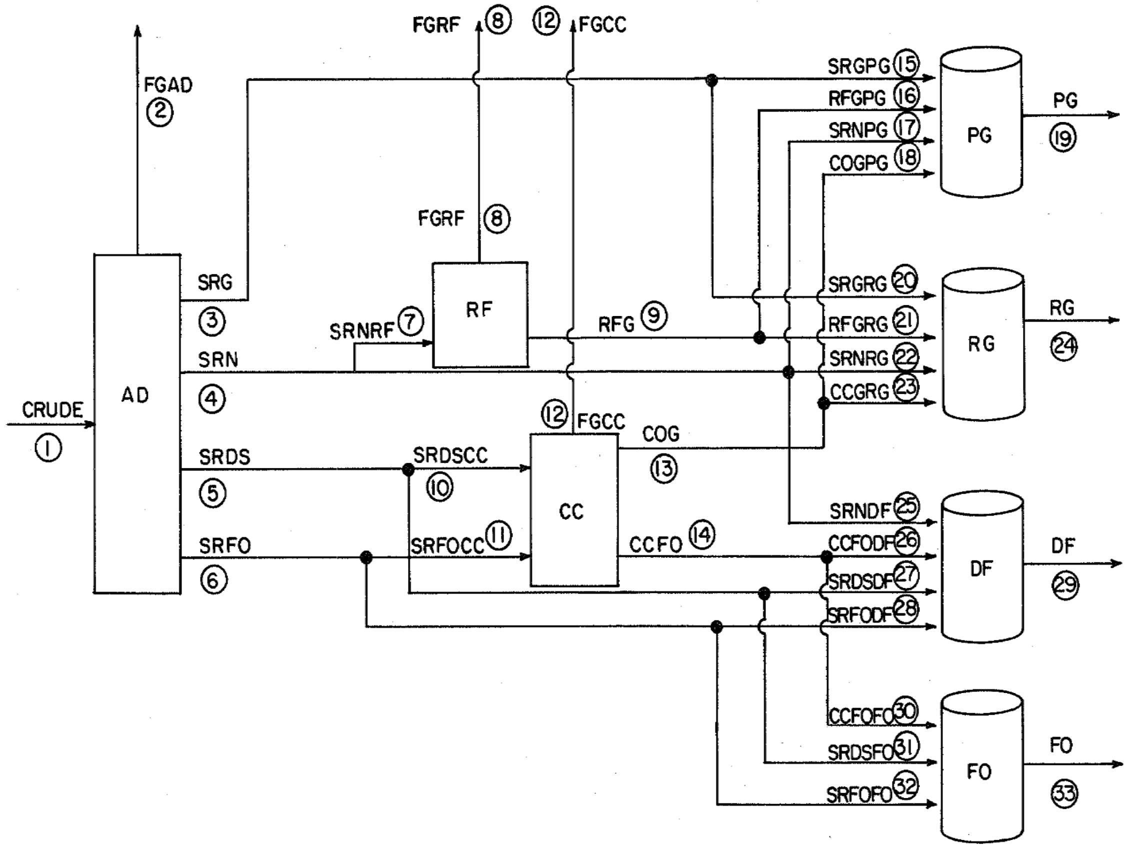

Figure 3-7 Process Flow Diagram for a Simple Refinery

Table 3-2 Definitions of the Names of the Process Streams for the Simple Petroleum Refinery

No.	Name	Definition (Flow rates are in barrels per day)
1	CRUDE	Crude oil flow rate to the atmospheric crude distillation column (AD)
2	FGAD	Fuel gas flow rate from AD
3	SRG	Straight run gasoline flow rate from AD
4	SRN	Straight run naphtha flow rate from AD
5	SRDS	Straight run distillate flow rate from AD
6	SRFO	Straight run fuel oil flow rate from AD
7	SRNRF	Straight run naphtha feed rate to the reformer (RF)
8	FGRF	Fuel gas flow rate from the reformer
9	RFG	Reformer gasoline flow rate
10	SRDSCC	Straight run distillate flow rate to the catalytic cracking unit (CCU)
11	SRFOCC	Straight run fuel oil flow rate to the CCU
12	FGCC	Fuel gas flow rate from the CCU
13	CCG	Gasoline flow rate from CCU
14	CCFO	Fuel oil flow rate from CCU
15	SRGPG	Straight run gasoline flow rate for premium gasoline (PG) blending
16	RFGPG	Reformer gasoline flow rate for PG blending
17	SRNPG	Straight run naphtha flow rate for PG blending
18	CCGPG	CCU gasoline flow rate for PG blending
19	PG	Premium gasoline flow rate
20	SRGRG	Straight run gasoline flow rate for regular gasoling (RG) blending
21	RFGRG	Reformer gasoline flow rate for RG blending
22	SRNRG	Straight run naphtha flow rate for RG blending
23	CCGRG	CCU gasoline flow rate for RG blending
24	RG	Regular gasoline flow rate
25	SRNDF	Straight run naphtha flow rate for diesel fuel (DF) blending
26	CCFODF	CCU fuel oil flow rate for DF blending
27	SRDSDF	Straight run distillate flow rate for Df blending
28	SRFODF	Straight run fuel oil flow rate for DF blending
29	DF	No. 2 diesel fuel flow rate
30	CCFOFO	CCU fuel oil flow rate for fuel oil (FO) blending
31	SRDSFO	Straight run distillate flow rate for FO blending
32	SRFOFO	Straight run fuel oil flow rate for FO blending
33	FO	No. 6 fuel oil flow rate

In Table 3-3 the capacities, operating costs, process stream, mass yields, and volumetric yields are listed for the three process units in the refinery. These are typical of a medium size refinery in the Gulf coast area. The mass yields were taken from those reported by Aronfsky, Dutton and Tayyaabkhan (10) and were converted to volumetric yields by using API gravity data. The operating costs were furnished by the technical division of a major oil company that has refineries on the Gulf Coast.

Table 3-3 Capacities, Operating Costs and Volumetric Yields for the Refinery Process Units

Unit	Capacity (bbl/day)	Operating Cost ($/bbl)	Input	Output	Mass Yield of Output Streams (1b/1b)	Volumetric Yields of Output Stream (bbl/bbl)
Crude Oil	100,000	1.00	CRUDE	FGAD	0.029	35.42
Atmospheric				SRG	0.236	0.270
Distillation				SRN	0.223	0.237
Column				SRDS	0.087	0.086
				SRFO	0.426	0.372
Catalytic	25,000	2.50	SRNRF	FGRF	0.138	158.7
Reformer				RFG	0.862	0.928
Catalytic	30,000	2.20	SRDSCC	FGCC	0.273	336.9
Cracking				CCG	0.536	0.619
Unit				CCFO	0.191	0.189
			SRFOCC	FGCC	0.277	386.4
				CCG	0.527	0.688
				CCFO	0.196	0.220

The quality specification and physical properties are given in Table 3-4 for the process streams, and the crude oil cost and the product sales prices are given in Table 3-5. The data in Table 3-4 was reported by Aronfsky et.al. (29), and the cost and prices in Table 3-5 were obtained from the *Oil and Gas Journal* (11). The information given in Table 3-3, 3-4, and 3-5 is required to construct the objective function and the constraint equations for the linear programming model of the refinery.

Table 3-4 Quality Specifications and Physical Properties for Products and Intermediate Streams for the Refinery

Stream	Motor Octane Number	Vapor Pressure (mm Hg)	Density (1b/bbl)	Sulfur Content (1b/bbl)
Premium Gasoline	$\geq$ 93.0	$\leq$ 12.7	-	-
Regular Gasoline	$\geq$ 87.0	$\leq$ 12.7	-	-
Diesel Fuel	-	-	$\leq$ 306.0	$\leq$ 0.5
Fuel Oil	-	-	$\leq$ 352.0	$\leq$ 3.0
SRG	78.5	18.4	-	-
RFG	104.0	2.57	-	-
SRN	65.0	6.54	272.0	0.283
CCG	93.7	6.90	-	-
CCFO	-	-	294.4	0.353
SRDS	-	-	292.0	0.526
SRFO	-	-	295.0	0.980

Table 3-5 Crude Oil Cost and Product Sales Prices for the Petroleum Refinery

Gulf Coast crude oil	\$32.00 / bbl
Premium gasoline	\$45.36 / bbl
Regular gasoline	\$43.68 / bbl
No. 2 diesel fuel	\$40.32 / bbl
No. 6 fuel oil	\$13.14 / bbl
Fuel gas	\$0.01965 / bbl or \$3.50 MSCF

It is standard practice to present the linear programming problem for the refinery in matrix form as shown in Figure 3-8. In the first row the coefficients of the terms in the objective function are listed under their corresponding variables. The sales prices are shown as positive, and the cost are shown as negative, so the problem is formulated to maximize the profit. These numbers were taken from Table 3-5, and it was convenient to combine the crude cost (\$32.00/Barrel) with the operating cost of the crude oil atmospheric distillation column (\$1.00/barrell) to show a total cost of \$33.00 per barrel of crude oil processed in Figure 3-8. Consequently, the first row of Figure 3-8 represents the objective function given below:

-33.0 CRUDE + 0.01965 FGAD - 2.50 SRNRF + 0.01965 FGRF - 2-20 SRDSCC
-2.20 SRFOCC + 0.01965 FGCC + 45.36 PG + 43.68 RG + 40.32 DF + 13.14 FO

The constraint equations begin with the second row in Figure 3-8. They are grouped in terms of quality and quantity constraints on the crude oil and products, in terms of the performance of the process unit using the volumetric yields, and in terms of the stream splits among the process units and blending into the products.

	Atmospheric Distillation						Reformer		
	CRUDE	FGAD	SRG	SRN	SRDS	SRFO	SRNRF	FGRF	RFG
Objective Function	-33.0	.01965					-2.50	.01965	
Crude Availability	1.0								
Products									
Premium Gasoline									
Min. PG Prod.									
PG Blending									
PG Octane Rating									
PG Vapor Press.									
Regular Gasoline									
Min. RG Prod.									
RG Blending									
RG Octane Rating									
RG Vapor Press.									
Diesel Fuel									
Min. DF Prod.									
DF Blending									
DF Density Spec.									
DF Sulfur Spec.									
Fuel Oil									
Min. FO Prod.									
FO Blending									
FO Density Spec.									
FO Sulfur Spec.									
Process Units									
Atm. Distillation									
AD Capacity	1.0								
FGAD Yield	35.42	-1.0							
SRG Yield	0.270		-1.0						
SRN Yield	0.237			-1.0					
SRDS Yield	0.087				-1.0				
SRFO Yield	0.372					-1.0			
Reformer									
RF Capacity							1.0		
FGRF Yield							158.7	-1.0	
RFG Yield							0.928		-1.0
Catalytic Cracker									
CC Capacity									
FGCC Yield									
CCG Yield									
CCFO Yield									
Stream Splits									
SRG			1.0						
SRN				1.0			-1.0		
SRDS					1.0				
SRFO						1.0			
RFG									1.0
CCG									
CCFO									

	Catalytic Cracker					Premium Gasoline Blending				
	SRDSCC	SRFOCC	FGCC	CCG	CCFO	SRGPG	RFGPG	SRNPG	CCGPG	PG
Objective Function	-2.20	-2.20	.01965							45.36
Crude Availability										
Products										
Premium Gasoline										
Min. PG Prod.										1.0
PG Blending						1.0	1.0	1.0	1.0	-1.0
PG Octane Rating						78.5	104.0	65.0	93.7	-93.0
PG Vapor Press.						18.4	2.57	6.54	6.90	-12.7
Regular Gasoline										
Min. RG Prod.										
RG Blending										
RG Octane Rating										
RG Vapor Press.										
Diesel Fuel										
Min. DF Prod.										
DF Blending										
DF Density Spec.										
DF Sulfur Spec.										
Fuel Oil										
Min. FO Prod.										
FO Blending										
FO Density Spec.										
FO Sulfur Spec.										
Process Units										
Atm. Distillation										
AD Capacity										
FGAD Yield										
SRG Yield										
SRN Yield										
SRDS Yield										
SRFO Yield										
Reformer										
RF Capacity										
FGRF Yield										
RFG Yield										
Catalytic Cracker										
CC Capacity	1.0	1.0								
FGCC Yield	336.9	386.4	-1.0							
CCG Yield	0.619	0.688		-1.0						
CCFO Yield	0.189	0.220			-1.0					
Stream Splits										
SRG						-1.0				
SRN								-1.0		
SRDS	-1.0									
SRFO		-1.0								
RFG							-1.0			
CCG				1.0					-1.0	
CCFO					-1.0					

	Regular Gasoline Blending					Diesel Fuel Blending				
	SRGRG	RFGRG	SRNRG	CCGRG	RG	SRNDF	CCFODF	SRDSDF	SRFODF	DF
Objective Function					43.68					40.32
Crude Availability										
Products										
Premium Gasoline										
Min. PG Prod.										
PG Blending										
PG Octane Rating										
PG Vapor Press.										
Regular Gasoline										
Min. RG Prod.					1.0					
RG Blending	1.0	1.0	1.0	1.0	-1.0					
RG Octane Rating	78.5	104.0	65.0	93.7	-87.0					
RG Vapor Press.	18.4	2.57	6.54	6.90	-12.7					
Diesel Fuel										
Min. DF Prod.										1.0
DF Blending						1.0	1.0	1.0	1.0	-1.0
DF Density Spec.						272.0	294.4	292.0	295.0	-306.0
DF Sulfur Spec.						0.283	0.383	0.526	0.980	-0.50
Fuel Oil										
Min. FO Prod.										
FO Blending										
FO Density Spec.										
FO Sulfur Spec.										
Process Units										
Atm. Distillation										
AD Capacity										
FGAD Yield										
SRG Yield										
SRN Yield										
SRDS Yield										
SRFO Yield										
Reformer										
RF Capacity										
FGRF Yield										
RFG Yield										
Catalytic Cracker										
CC Capacity										
FGCC Yield										
CCG Yield										
CCFO Yield										
Stream Splits										
SRG	-1.0									
SRN			-1.0			-1.0				
SRDS								-1.0		
SRFO									-1.0	
RFG		-1.0								
CCG				-1.0						
CCFO							-1.0			

	Fuel Oil Blending					
	CCFOFO	SRDSFO	SRFOFO	FO		
Objective Function				13.14	= Maximum	OBJ
Crude Availability					≤110,000	CRDAVAIL
Products						
Premium Gasoline						
Min. PG Prod.					≥10,000	PGMIN
PG Blending					= 0	PGBLEND
PG Octane Rating					≥ 0	PGOCTANE
PG Vapor Press.					≤ 0	PGVAPP
Regular Gasoline						
Min. RG Prod.					≥10,000	RGMIN
RG Blending					= 0	RGBLEND
RG Octane Rating					≥ 0	RGOCTANE
RG Vapor Press.					≤ 0	RGVAPP
Diesel Fuel						
Min. DF Prod.					≥ 10,000	DFMIN
DF Blending					= 0	DFBLEND
DF Density Spec.					≤ 0	DFDENS
DF Sulfur Spec.					≤ 0	DFSULFUR
Fuel Oil						
Min. FO Prod.				1.0	≥ 10,000	FOMIN
FO Blending	1.0	1.0	1.0	-1.0	= 0	FOBLEND
FO Density Spec.	294.4	292.0	295.0	-352.0	≤ 0	FODENS
FO Sulfur Spec.	0.353	0.526	0.980	-3.0	≤ 0	FOSULFUR
Process Units						
Atm. Distillation						
AD Capacity					≤100,000	ADCAP
FGAD Yield					= 0	ADFGYLD
SRG Yield					= 0	ADSRGYLD
SRN Yield					= 0	ADNYLD
SRDS Yield					= 0	ADDSYLD
SRFO Yield					= 0	ADFOYLD
Reformer						
RF Capacity					≤ 25,000	RFCAP
FGRF Yield					= 0	RFFGYLD
RFG Yield					= 0	RFRFGYLD
Catalytic Cracker						
CC Capacity					≤ 30,000	CCCAP
FGCC Yield					= 0	CCFGYLD
CCG Yield					= 0	CCGYLD
CCFO Yield					= 0	CCFOYLD
Stream Splits						
SRG					= 0	SRGSPLIT
SRN					= 0	SRNSPLIT
SRDS		-1.0			= 0	SRDSSPLT
SRFO			-1.0		= 0	SRFOPLIT
RFG					= 0	RFGSPLIT
CCG					= 0	CCGSPLIT
CCFO	-1.0				= 0	CCFOSPLT

Figure 3-8 Refinery Objective Function and Constraint Equations

The second row is the crude availability constraint limiting the refinery to 110,000 barrels/day. This is followed by the four quantity and quality constraints associated with each product. These are the daily production and blending requirements and two quality constraints. These have been extracted from Figure 3-8 and are shown in Table 3-6 for the four products. The minimum production constraint states that the refinery must produce at least 10,000 barrels/day of premium gasoline to meet the company's marketing division's requirements. The blending constraints state that the sum of the streams going to produce premium gasoline must equal the daily production of premium gasoline. The quality constraints use linear blending, and the sum of each component weighted by its quality must meet or exceed the quality of the product. This is illustrated with premium gasoline octane rating blending constraint which is written as the following using the information from the matrix:

$$78.5\ \text{SRGPG} + 104.0\ \text{RFGPG} + 65.0\ \text{SRNPG} + 93.7\ \text{CCGPG} - 93.0\ \text{PG} \geq 0 \qquad (3\text{-}11)$$

Here the premium gasoline must have an octane number of at least 93.0. Corresponding, inequality constraints are specified in Table 3-6 using the same procedure for premium gasoline vapor pressure, regular gasoline octane number and vapor pressure, diesel fuel density and sulfur content and fuel oil density and sulfur content.

The next set of information given in the constraint equation matrix, Figure 3-8, is the description of the operation of the process unit using the volumetric yield shown in Table 3-3. This section of the matrix has been extracted and is shown in Table 3-7 for the three process units. Referring to the volumetric yields for the crude oil distillation column, these data states that 35.42 times the volumetric flow rate of crude produces the flow rate of fuel gas from the distillation column, FGAD, i.e.:

$$35.42\ \text{CRUDE} - \text{FGAD} = 0 \qquad (3\text{-}12)$$

Corresponding yields of the other products from crude oil distillation are determined the same way. For the catalytic reformer the yield of the fuel gas (FGRF) and the reformer gasoline (RFG) are given by the following equations:

$$158.7\ \text{SRNRF} - \text{FGRF} = 0 \qquad (3\text{-}13)$$

$$0.928\ \text{SRNRF} - \text{RFG} = 0 \qquad (3\text{-}14)$$

Similar equations are used in the matrix, Figure 3-8, and are summarized in Table 3-7 for the process units in the simple refinery.

Table 3-6 Quantity and Quality Constraints for the Refinery Products

Premium Gasoline

	SRGPG	RFGPG	SRNPG	CCGPG	PG	RHS
Min. P.G. Production					1.0	≥ 1,000
PG blending	1.0	1.0	1.0	1.0	- 1.0	= 0
PG octane rating	78.5	104.0	65.0	93.7	- 93.0	≥ 0
PG vapor pressure	18.4	2.57	6.54	6.90	- 12.7	≤ 0

Regular Gasoline

	SRGRG	RFGRG	SRNRG	CCGRG	RG	RHS
Min R.G. production					1.0	≤ 10,000
RG blending	1.0	1.0	1.0	1.0	- 1.0	= 0
RG octane rating	78.5	104.0	65.0	93.7	- 87.0	≤ 0
RG vapor pressure	18.4	2.57	6.54	6.90	- 12.7	≤ 0

Diesel Fuel

	SRNDF	CCFODF	SRDSDF	SRFODF	DF	RHS
Min D.F. production					1.0	≥ 10,000
DF blending	1.0	1.0	1.0	1.0	- 1.0	= 0
DF density spec.	272.0	294.4	292.0	295.0	- 306.0	≤ 0
DF sulfur spec.	0.283	0.353	0.526	0.980	- 0.50	≤ 0

Fuel Oil

	CCFOFO	SRDSFO	SRFOFO	FO	RHS
Min. FO production				1.0	≥ 10,000
FO blending	1.0	1.0	1.0	- 1.0	= 0
FO density spec.	294.4	292.0	295.0	- 352.0	≤ 0
FO sulfur spec.	0.353	0.526	0.980	- 3.0	≤ 0

The use of volumetric yields to give linear equations to describe the performance of the process units is required for linear programming. The results will be satisfactory as long as the volumetric yields precisely describe the performance of these process units. These volumetric yields are a function of the operating conditions of the unit, e.g. temperature, feed flow rate, catalyst activity, etc. Consequently, to have an optimal solution these volumetric yields must represent the best performance of the individual process units. To account for changes in volumetric yields with operating conditions sometimes a separate simulation program is coupled to the linear programming code to furnish best values of the volumetric yields. Then an iterative procedure is used to converge to the optimal operating conditions with corresponding values of volumetric yields from the simulation program. (See Figure 4-5.)

Table 3-7 Process Unit Material Balances using Volumetric Yields

Crude oil atmospheric distillation column

	CRUDE	FGAD	SRG	SRN	SRDS	SRFO	RHS
AD Capacity	1.0						≤ 100,000
FGAD Yield	35.42	- 1.0					= 0
SRG Yield	0.270		- 1.0				= 0
SRN Yield	0.237			- 1.0			= 0
SRDS Yield	0.086				- 1.0		= 0
SRFO Yield	0.372					- 1.0	= 0

Catalytic reformer

	SRNRF	FGRF	RFG	RHS
RF Capacity	1.0			≤ 25,000
FGRF Yield	158.7	- 1.0		= 0
RFG Yield	0.928		- 1.0	= 0

Catalytic cracking unit

	SRDSCC	SRFOCC	FGCC	CCG	CCFO	RHS
CC Capacity	1.0	1.0				≤ 30,000
FGCC Yield	336.9	86.4	- 1.0			= 0
CCG Yield	0.619	0.688		- 1.0		= 0
CCFO Yield	0.189	0.220			- 1.0	= 0

The last group of terms in Figure 3-8 gives the material balance around points where streams split among process units and blend into products. The stream to be divided is given a coefficient of one, and the resulting streams have a coefficient minus one. For example, the straight run naphtha from the crude oil distillation is split into four streams. One is sent to the catalytic reformer and the other three are used in blending premium gasoline, regular gasoline and diesel fuel. The equation for this split is:

$$\text{SRN} - \text{SRNRF} - \text{SRNPG} - \text{SRNRG} - \text{SRNDF} = 0 \qquad (3\text{-}15)$$

There is a total of seven stream splits as shown in Figure 3-8.

The information is now available to determine the optimum operating conditions of the refinery. There are 83 independent variables, and 38 constraint equations (23 equality constraints and 15 equality constraints). The optimal solution was obtained using the Mathematical Programming System Extended **(**MPSX**)** program run on the IBM 4341. The format used by this linear programming code has become an industry standard according to Murtagh (12) and is not restricted to the MPS series of codes developed originally for IBM computers. Consequently, we will also describe the input procedure for the code because of its more general nature. Also, we will use these refinery results to illustrate the additional information that can be obtained from sensitivity analysis. Similar, but not as detailed, results can be obtained using Excel.

Solving the Linear Programming Problem for the Simple Refinery

Having constructed the linear programming problem matrix, we are now ready to solve the problem using a large linear programming computer program. The input and output for these programs has become relatively standard (12) making the study of one beneficial in the use of any of the others. The solution of the simple refinery has been obtained using the IBM Mathematical Programming System Extended (MPSX). The detailed documentation is given in IBM manuals (15, 16) and by Murtagh (12) on the use of the program, and the following outlines its use for the refinery problem. The MPSX control program used to solve the problem is given in Table 3-8. The first two commands, PROGRAM and INITIALZ, define the beginning of the program and set up standard default values for many of the optional program parameters. TITLE writes the character string between the quotation marks at the top of every page of output. The four MOVE commands give user specified names to the input data (XDATA), internal machine code version of the problem (XPBNAME), objective function (XOBJ), and right-hand-side vector (XRHS). Next, CONVERT calls a routine to convert the input data from binary coded decimal (BCD) or communications format into machine code for use by the program, and BCDOUT has the input data printed. The next three commands, SETUP, CRASH and PRIMAL, indicate that the objective function is to be maximized, a starting basis is created, and the primal method is to be used to solve the problem. Output from PRIMAL is in machine code so SOLUTION is called to produce BCD output of the solution. The RANGE command is used in the sensitivity analysis to determine the range over which the variables, right-hand-sides and the coefficients may vary without changing the basis. The last two statements, EXIT and PEND, signal the end of the control program and return control over to the computer's operating system.

Input to the MPSX program is divided into four sections: NAME, ROWS, COLUMNS, and RHS. The first two are shown in Table 3-9. The NAME section is a single line containing the identifier, NAME, and the user-defined name for the block of input data that follows. (MPSX has provisions for keeping track of several problems during execution of the control program). When the program is run it looks for input data with the same name as that stored in the internal variable XDATA. The ROWS section contains the name of every row in the model, preceded by a letter indicating whether it is a non-constrained row (N), the objective function, a less-than-or-equal-to constraint (L), a greater-than-or-equal-to constraint (G), or an equality constraint (E).

The COLUMNS section of the input data is shown in Table 3-10. It is a listing of the non-zero elements in each column of the problem matrix (Figure 3-8). Each line contains a column name followed by up to two row names and their corresponding coefficients from Figure 3-8.

The last input section is shown in Table 3-11. Here, the right- hand-side coefficients are entered in the same way that the coefficients for each column were entered in the COLUMNS section, i.e., only the non-zero elements. The end of the data block is followed by an ENDATA card.

The solution to the refinery problem is presented in Table 3-12 (a) and (b) as listed in the printout from the MPSX program. It is divided into two sections, the first providing information about the constraints (rows) and the second giving information about the refinery stream variables (columns).

Table 3-8 Mathematical Programming System Control Program for the Simple Refinery

```
PROGRAM
INITIALZ
TITLE('SIMPLE REFINERY MODEL')
MOVE(XDATA,'REFINERY')
MOVE(XPBNAME,'REFINERY')
MOVE(XOBJ,'OBJ')
MOVE(XRHS,'RHS')
CONVERT('SUMMARY')
BCDOUT
SETUP('MAX')
PICTURE
CRASH
PRIMAL
SOLUTION
RANGE
EXIT
PEND
```

Table 3-9 MPSX Input NAME and ROWS Sections

```
NAME          REFINERY
ROWS
 N      OBJ
 L      CRDAVAIL
 G      PGMIN
 E      PGBLEND
 G      PGOCTANE
 L      PGVAPP
 G      RGMIN
 E      RGBLEND
 G      RGOCTANE
 L      RGVAPP
 G      DFMIN
 E      DFBLEND
 L      DFDENS
 L      DFSULFUR
 G      FOMIN
 E      FOBLEND
 L      FODENS
 L      FOSULFUR
 L      ADCAP
 E      ADFGYLD
 E      ADSRGYLD
 E      ADNYLD
 E      ADDSYLD
 E      ADFOYLD
 L      RFCAP
 E      RFFGYLD
 E      RFRFGYLD
 L      CCCAP
 E      CCFGYLD
 E      CCGYLD
 E      CCFOYLD
 E      SRGSPLIT
 E      SRNSPLIT
 E      SRDSSPLT
 E      SRFOSPLT
 E      RFGSPLIT
 E      CCGSPLIT
 E      CCFOSPLT
```

Table 3-10 MPSX Input COLUMNS Section

COLUMNS

CRUDE	OBJ	-33.0	CRDAVAIL	1.0
CRUDE	ADCAP	1.0	ADFGYLD	35.42
CRUDE	ADSRGYLD	0.270	ADNYLD	0.237
CRUDE	ADDSYLD	0.087	ADFOYLD	0.372
FGAD	OBJ	0.01965	ADFGYLD	-1.0
SRG	ADSRGYLD	-1.0	SRGSPLIT	1.0
SRN	ADNYLD	-1.0	SRNSPLIT	1.0
SRDS	ADDSYLD	-1.0	SRDSSPLT	1.0
SRFO	ADFOYLD	-1.0	SRFOSPLT	1.0
SRNRF	OBJ	-2.50	RFCAP	1.0
SRNRF	RFFGYLD	158.7	RFRFGYLD	0.928
SRNRF	SRNSPLIT	-1.0		
FGRF	OBJ	0.01965	RFFGYLD	-1.0
RFG	RFRFGYLD	-1.0	RFGSPLIT	1.0
SRDSCC	OBJ	-2.20	CCCAP	1.0
SRDSCC	CCFGYLD	336.9	CCGYLD	0.619
SRDSCC	CCFOYLD	0.189	SRDSSPLT	-1.0
SRFOCC	OBJ	-2.20	CCCAP	1.0
SRFOCC	CCFGYLD	386.4	CCGYLD	0.688
SRFOCC	CCFOYLD	0.2197	SRFOSPLT	-1.0
FGCC	OBJ	0.01965	CCFGYLD	-1.0
CCG	CCGYLD	-1.0	CCGSPLIT	1.0
CCFO	CCFOYLD	-1.0	CCFOSPLT	1.0
SRGPG	PGBLEND	1.0	PGOCTANE	78.5
SRGPG	PGVAPP	18.4	SRGSPLIT	-1.0
RFGPG	PGBLEND	1.0	PGOCTANE	104.0
RFGPG	PGVAPP	2.57	RFGSPLIT	-1.0
SRNPG	PGBLEND	1.0	PGOCTANE	65.0
SRNPG	PGVAPP	6.54	SRNSPLIT	-1.0
CCGPG	PGBLEND	1.0	PGOCTANE	93.7
CCGPG	PGVAPP	6.90	CCGSPLIT	-1.0
PG	OBJ	45.36	PGMIN	1.0
PG	PGBLEND	-1.0	PGOCTANE	-93.0
PG	PGVAPP	-12.7		
SRGRG	RGBLEND	1.0	RGOCTANE	78.5
SRGRG	RGVAPP	18.4	SRGSPLIT	-1.0
RFGRG	RGBLEND	1.0	RGOCTANE	104.0
RFGRG	RGVAPP	2.57	RFGSPLIT	-1.0
SRNRG	RGBLEND	1.0	RGOCTANE	65.0

Table 3-10 MPSX Input COLUMNS Section (continued)

CCFODF	DFBLEND	1.0	DFDENS	294.4
CCFODF	DFSULFUR	0.353	CCFOSPLT	-1.0
SRDSDF	DFBLEND	1.0	DFDENS	292.0
SRDSDF	DFSULFUR	0.526	SRDSSPLT	-1.0
SRFODF	DFBLEND	1.0	DFDENS	295.0
SRFODF	DFSULFUR	0.98	SRFOSPLT	-1.0
DF	OBJ	40.32	DFMIN	1.0
DF	DFBLEND	-1.0	DFDENS	-306.0
DF	DFSULFUR	-0.5		
CCFOFO	FOBLEND	1.0	FODENS	294.4
CCFOFO	FOSULFUR	0.353	CCFOSPLT	-1.0
SRDSFO	FOBLEND	1.0	FODENS	292.0
SRDSFO	FOSULFUR	0.526	SRDSSPLT	-1.0
SRFOFO	FOBLEND	1.0	FODENS	295.0
SRFOFO	FOSULFUR	0.98	SRFOSPLT	-1.0
FO	OBJ	13.14	FOMIN	1.0
FO	FOBLEND	-1.0	FODENS	-352.0
FO	FOSULFUR	-3.00		

Table 3-11 MPSX Input Right Hand Side Section

RHS

RHS	CRDAVAIL	110000.0	PGMIN	10000.0
RHS	RGMIN	10000.0	DFMIN	10000.0
RHS	FOMIN	10000.0	ADCAP	100000.0
RHS	RFCAP	25000.0	CCCAP	30000.0

ENDATA

In the ROWS section of Table 3-12(a) there are eight columns of output. The first is the internal identification number given to each row by the program. The second column is the name given to the rows in the input data. Next is the AT column which contains a pair of code letters to indicate the status of each row in the optimal solution. Constraint rows in the basis have the code BS, non-basis inequality constraints that have reached their upper or lower limits have the code UL or LL. Equality constraints have the status code EQ. The fourth column is the row activity, as defined by the equation:

$$\text{Activity}_{\text{i}} = \sum_{j=1}^{m} a_{ij} x_j$$

This is the optimal value of the left-hand side of the constraint equations. However, it is computed by subtracting the slack variable from the right-hand side. The column labeled SLACK ACTIVITY contains the value of the slack variable for each row. The next three columns are associated with sensitivity analysis. The sixth and seventh columns show the lower and upper limits placed on the row activities. The final column, DUAL ACTIVITY, gives Lagrange multipliers that are also called the *simplex multipliers, shadow prices* and *implicit prices*. As will be seen subsequently in sensitivity analysis, they will relate changes in the activity to changes in the objective function. Also, the dot in the table means zero, the same convention used in the Simplex Tableau.

Examination of this section of output shows that the activity (or value) of the objective function (row 1, OBJ) is 701,823.4, i.e., the maximum profit for the refinery is $701,823.40 per day. Checking the rows which are at their lower limits, LL, for production constraints one finds that only row 15, FOMIN, is at its lower limit of 10,000 bbl/day indicating that only the minimum required amount of fuel oil should be produced. However, row 3, PGMIN, row 7, RGMIN, and row 11, DFMIN, are all above their lower limits with values of 47,113 bbl/day for premium gasoline, 22,520 bbl/day for regular gasoline, and 12,491 bbl/day for diesel fuel. More will be said about the information in this table when sensitivity analysis is discussed.

The COLUMNS section of Table 3-12(b) for the optimal solution also has eight columns. The first three are analogous to the first three in the ROWS section, i.e., an interval identification number, name of the column, and whether the variable is in the basis BS or is at its upper or lower limit, UL or LL. The fourth column, ACTIVITY, contains the optimal value for each variable. The objective function cost coefficients are listed in the column INPUT COST. REDUCED COST is the amount by which the objective function will be increased per unit increase in each non-basis variable and is part of the sensitivity analysis. It is given by c_j' of Equation (4-29).

For this simple refinery model there were 33 variables whose optimal value were determined, and 38 constraint equations were satisfied. For an actual refinery there would be thousands of constraint equations, but they would be developed in the same fashion as described here. As can be seen, the model (constraint equations) was simple, and only one set of operating conditions was considered for the catalytic cracking unit, catalytic reformer and the crude distillation column.

If the optimal flow rates do not match the corresponding values for volumetric yields, a search can be performed by repeating the problem to obtain a match of the optimal flow rates and volumetric yields. This has to be performed using a separate simulation program that generates volumetric yields from flow rate through the process units. (See Figure 3-5). Thus, the linear model of the plant can be made to account for nonlinear process operations. Another procedure, successive (or sequential) linear programming uses linear programming iteratively, also; and it will be discussed in Chapter 5. The state of industrial practice using both linear programming and successive linear programming is described by Smith and Bonner (13) for configuration of new refineries and chemical plants, plant expansions, economic evaluation of investment alternatives, assessment of new technology, operating plans for existing plants, variation in feeds, costing and distribution of products, evaluation of processing and exchange agreements, forecasting of industry trends and economic impact of regulatory changes.

Table 3-12(a) MPSX Output for Optimal Solution, Section 1 - Rows

NUMBER	ROW	AT	ACTIVITY	SLACK ACTIVITY	LOWER LIMIT	UPPER LIMIT	DUAL ACTIVITY
1	OBJ	BS	701823.4	-701823.4	NONE	NONE	1.000
2	CRDAVAIL	BS	100000.0	10000.0	NONE	10000.0	.
3	PGMIN	BS	47113.2	-37113.2	10000.0	NONE	.
4	PGBLEND	EQ	.	.	.	.	19.320
5	PGOCTANE	LL	.	.	.	NONE	0.280
6	PGVAPP	BS	-188607.2	188607.2	NONE	.	.
7	RGMIN	BS	22520.4	12520.4	10000.0	NONE	.
8	RGBLEND	EQ	.	.	.	.	19.320
9	RGOCTANE	LL	.	.	.	NONE	0.280
10	RGVAPP	UL	.	.	NONE	.	.
11	DFMIN	BS	12491.0	-2491.0	10000.0	NONE	.
12	DFBLEND	EQ	.	.	.	.	40.320
13	DFDENS	BS	-165458.8	165458.8	NONE	.	.
14	DFSULFUR	UL	.	.	NONE	.	.
15	FOMIN	LL	10000.0	.	10000.0	NONE	27.180
16	FOBLEND	EQ	.	.	.	.	40.320
17	FODENS	BS	-571996.8	571996.8	NONE	.	.
18	FOSULFUR	BS	-22286.7	22286.7	NONE	.	.
19	ADCAP	UL	100000.0	.	NONE	100000.0	-8.154
20	ADFGYLD	EQ	.	.	.	.	0.01965
21	ADSRGYLD	EQ	.	.	.	.	41.300
22	ADNYLD	EQ	.	.	.	.	45.571
23	ADDSYLD	EQ	.	.	.	.	40.320
24	ADFOYLD	EQ	.	.	.	.	40.320
25	RFCAP	BS	23700.0	1300.0	NONE	25000.0	.
26	FGRFYLD	EQ	.	.	.	.	0.01965
27	RFRFGYLD	EQ	.	.	.	.	48.440
28	CCCAP	UL	30000.0	.	NONE	30000.0	5.274
29	CCFGYLD	EQ	.	.	.	.	0.01965
30	CCGYLD	EQ	.	.	.	.	45.5560
31	CCFOYLD	EQ	.	.	.	.	40.3200
32	SRGSPLIT	EQ	.	.	.	.	41.3000
33	SRNSPLIT	EQ	.	.	.	.	45.5708
34	SRDSSPLT	EQ	.	.	.	.	40.320
35	SRFOSPLT	EQ	.	.	.	.	40.320
36	RFGSPLIT	EQ	.	.	.	.	48.440
37	CCGSPLIT	EQ	.	.	.	.	45.556
38	CCFOSPLT	EQ	.	.	.	.	40.320

Table 3-12(b) MPSX Output for Optimal Solution, Section 2 - Columns

NUMBER	COLUMN	AT	ACTIVITY	INPUT COST	LOWER LIMIT	UPPER LIMIT	REDUCED COST
39	CRUDE	BS	100000.0	-33.00	.	NONE	.
40	FGAD	BS	3542000.0	0.01965	.	NONE	.
41	SRG	BS	27000.0	.	.	NONE	.
42	SRN	BS	23700.0	.	.	NONE	.
43	SRDS	BS	8700.0	.	.	NONE	.
44	SRFO	BS	37200.0	.	.	NONE	.
46	FGRF	BS	761190.0	0.01965	.	NONE	.
47	RFG	BS	21993.6	.	.	NONE	.
48	SRDSCC	LL	.	-2.20	.	NONE	-5.354
49	SRFOCC	BS	30000.0	-2.20	.	NONE	.
50	FGCC	BS	11592000.0	0.01965	.	NONE	.
51	CCG	BS	20640.0	.	.	NONE	.
52	CCFO	BS	6591.0	.	.	NONE	.
53	SRGPG	BS	13852.0	.	.	NONE	.
54	RFGPG	BS	17240.0	.	.	NONE	.
55	SRNPG	LL	.	.	.	NONE	-8.051
56	CCGPG	BS	16021.1	.	.	NONE	.
57	PG	BS	47113.2	45.36	.	NONE	.
58	SRGRG	BS	13148.0	.	.	NONE	.
59	RFGRG	BS	4753.6	.	.	NONE	.
60	SRNRG	LL	.	.	.	NONE	-8.051
61	CCGRG	BS	4618.8	.	.	NONE	.
62	RG	BS	22520.4	43.68	.	NONE	.
63	SRNDF	LL	.	.	.	NONE	-5.251
64	CCFODF	BS	3263.0	.	.	NONE	.
65	SRDSDF	BS	8700.0	.	.	NONE	.
66	SRFODF	BS	528.0	.	.	NONE	.
67	DF	BS	12491.0	40.32	.	NONE	.
68	CCFOFO	BS	3328.0	.	.	NONE	.
69	SRDSFO	LL	.	.	.	NONE	.
70	SRFOFO	BS	6672.0	.	.	NONE	.
71	FO	BS	10000.0	13.14	.	NONE	.

Sensitivity Analysis

Having obtained the optimal solution for a linear programming problem, it would be desirable to know how much the cost coefficients could change, for example, before it is necessary to resolve the problem. In fact, there are five areas that should be examined for their effect on the optimal solution. These are:

1. Changes in the right-hand side of the constraint equations, b_i.
2. Changes in the coefficients of the objective function, c_j.
3. Changes in the coefficients of the constraint equations, a_{ij}.
4. Addition of new variables.
5. Addition of more constraint equations.

Changes in the right-hand side of the constraint equations correspond to changes in the maximum capacity of a process unit or the availability of a raw material, for example. Changes in the coefficients of the objective function correspond to changes of the cost or the sale price of the raw materials and products. Changes in the coefficients of the constraint equations correspond to changes in volumetric yields of a process. Addition of new variables and constraint equations correspond to the addition of new process units in the plant. It is valuable to know how these various coefficients and parameters can vary without changing the optimal solution, and this may reduce the number of times the linear programming problem must be solved.

Prior to doing this post-optimal analysis some preliminary mathematical expressions must be developed for the analysis of the effect of the above five areas on the optimal solution. These are the inverse of the optimal basis and the Lagrange multipliers. To obtain the matrix called the inverse of the optimal basis, $\mathbf{A}^{*-1}$, consider that the optimal basis has been found by the previously described Simplex Method. There are m constraint equations and n variables as given by Equations 3-1a, b and c. For convenience, the nonzero variables in the optimal basis have been rearranged to go from 1 to m, $(x_1^*, x_2^* ..., x_m^*, 0, ..., 0)$; and there are $(n - m)$ variables not in the basis whose value is zero. The optimal solution to this linear programming problem is indicated below where $\mathbf{x}^*$ contains only the m nonzero basis variables.

$$p^* = \mathbf{c}^T \mathbf{x}^* = \underset{\mathbf{x}}{opt}\ \mathbf{c}^T \mathbf{x} \tag{3-16}$$

and

$$\mathbf{A}^* \mathbf{x}^* = \mathbf{b} \tag{3-17}$$

To solve for $\mathbf{x}^*$, both sides of the above equation are multiplied by the inverse of the optimal basis, $\mathbf{A}^{*-1}$ whose elements are β_{ij} and obtain:

$$\mathbf{x}^* = \mathbf{A}^{*-1}\mathbf{b} \tag{3-18}$$

It should be noted that $\mathbf{A}^{*-1}$ may be obtained from the last step of the Simplex Method if all of the constraint equations required slack variables. If not, then it has to be obtained from the original formulation of the problem using the optimal basis found from the Simplex Method.

The linear programming problem could be solved by the classical method of Lagrange multipliers. However, the Simplex Method gives a systematic procedure for locating the optimal basis. Having located the optimal basis by the Simplex Method, the Lagrange multiplier formulation and the inverse of the optimal basis will be used to determine the effect of change in the right-hand side on the optimal solution. Consequently, it is necessary to compute the values of the Lagrange multipliers as follows. Multiplying each constraint Equation, (3-1b), by the Lagrange multiplier λ_i and adding to the objective function Equation (3-1a), gives the following equation.

$$\left[c_1 + \sum_{i=1}^{m} a_{i1}\lambda_i\right]x_1 + \left[c_2 + \sum_{i=1}^{m} a_{i2}\lambda_i\right]x_2 + \ldots + \left[c_m + \sum_{i=1}^{m} a_{im}\lambda_i\right]x_m + \left[c_{m+1} + \sum_{i=1}^{m} a_{i,m+1}\lambda_i\right]x_{m+1} + \ldots + \left[c_n + \sum_{i=1}^{m} a_{in}\lambda_i\right]x_n = p + \sum_{i=1}^{m} b_i\lambda i \qquad (3\text{-}19)$$

where x_1 to x_m are positive numbers i.e. values of the variables in the basis, and x_{m+1} to x_n are zero, i.e. values of the variables that are not in the basis.

To solve this problem by classical methods the partial derivatives of p with respect to the independent variables and the Lagrange multipliers would be set equal to zero. Taking the partial derivatives of p with respect to the Lagrange multipliers just gives the constraint equations, and taking the partial derivatives with respect to the independent variables, x_j^* (j = 1, 2, ...m) gives:

$$\frac{\partial p}{\partial x_j} = \left[c_j + \sum_{i=1}^{m} a_{ij} * \lambda_i\right] = 0 \quad \text{for j = 1, 2, …, m} \qquad (3\text{-}20)$$

and x_j^* for $j = m + 1, \ldots n$ is zero, since $\mathbf{x}^*$ is the optimal solution.

The values of the Lagrange multipliers are obtained from the solution of Equation (3-20). Written in matrix notation, Equation (3-20) is:

$$\mathbf{c} + \mathbf{A}^{*T}\boldsymbol{\lambda} = 0 \qquad (3\text{-}21)$$

where $\mathbf{A}^{*T}$ is the transpose of the matrix A*.

Using the matrix identity $[\mathbf{A}^{*T}]^{-1} = [\mathbf{A}^{*-1}]^T$ and solving for the Lagrange multipliers gives:

$$\boldsymbol{\lambda} = -[\mathbf{A}^{*-1}]^{\mathrm{T}}\,\mathbf{c} \qquad (3\text{-}22)$$

In terms of the elements of the inverse of the optimal basis β_{ij}, Equation (3-22) can be written as:

$$\lambda_i = -\sum_{j=1}^{m} \beta_{ij} c_j \quad \text{for i} = 1,2 \ldots, \text{m} \tag{3-23}$$

With this as background, the effect of the five changes on the optimal solution can be determined. The inverse of the optimal basis $\mathbf{A}^{*-1}$ and the Lagrange multipliers will be used to evaluate these changes. The following example illustrates the computation of the inverse of the optimal basis and the Lagrange multipliers.

Example 3-6

Solve the following problem by the Simplex Method and compute the inverse of the optimal basis and the Lagrange multipliers:

$$\begin{aligned} \text{maximize:} \quad & 2x_1 + x_2 + x_3 \\ \text{subject to:} \quad & x_1 + x_2 + x_3 \leq 10 \\ & x_1 + 5x_2 + x_3 \geq 20 \end{aligned}$$

Adding slack variables gives:

$$\begin{aligned} \text{maximize:} \quad & 2x_1 + x_2 + x_3 && = p \\ \text{subject to:} \quad & x_1 + x_2 + x_3 + x_4 && = 10 \\ & x_1 + 5x_2 + x_3 - x_5 && = 20 \end{aligned}$$

An initially feasible basis is not available, and either artificial variables or algebraic manipulations must be performed to obtain one. Algebraic manipulations are used to have x_1 and x_2 be the variables in the basis. The result is:

$$\begin{aligned} - x_3 - 9/4x_4 - 1/4x_5 &= p - 17½ \qquad & p &= 17½ \\ x_1 + x_3 + 5/4x_4 - 1/4x_5 &= 7½ & x_1 &= 7½ \\ + x_2 - 1/4x_4 - 1/4x_5 &= 2½ & x_2 &= 2½ \\ & & x_3 &= 0 \\ & & x_4 &= 0 \\ & & x_5 &= 0 \end{aligned}$$

This is the optimum since all of the coefficients of the non-basic variables in the objective function are negative. Knowing the optimal solution, the original problem now takes the form:

$$\begin{aligned} \text{maximize:} \quad & 2x_1 + x_2 = 17\tfrac{1}{2} \\ \text{subject to:} \quad & x_1 + x_2 = 10 \\ & x_1 + 5x_2 = 20 \end{aligned}$$

The inverse of the optimal basis is computed using the co-factor method.

$$\boldsymbol{A}^{*-1} = (-1)^{i+j} \frac{1}{|\boldsymbol{A}^*|} \|\boldsymbol{A}^*_{ji}\|$$

where $\|\mathbf{A}^*_{ji}\| = \|\mathbf{A}^*_{ij}\|^T$, and $\|\mathbf{A}^*_{ij}\|$ are the co-factors of the matrix $\mathbf{A}^*$.(8)

$$A^* = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} \quad |A^*| = 5 - 1 = 4 \quad \|A^*_{ji}\| = \begin{Vmatrix} 5 & -1 \\ -1 & 1 \end{Vmatrix}$$

$$A^{*-1} = \begin{bmatrix} 5/4 & -1/4 \\ -1/4 & 1/4 \end{bmatrix} \quad \text{and} \quad A^{*-1}A^* = \begin{bmatrix} 5/4 & -1/4 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The Lagrange multipliers are computed using Equation (3-22)

$$\boldsymbol{\lambda} = -[\mathbf{A}^{*-1}]^T \mathbf{c}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = -\begin{bmatrix} 5/4 & -1/4 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = -\begin{bmatrix} 9/4 \\ -1/4 \end{bmatrix}$$

or

$\lambda_1 = -9/4$ and $\lambda_2 = 1/4$

Changes in the Right-Hand Side of the Constraint Equations: Changes in the right-hand side of the constraint equations, i.e. changes in the b_i's, will cause changes in the values of the variables in the optimal solution, the x_j's. For an optimal solution to remain optimal, the x_j's cannot become negative. Equation (3-18) will be used to evaluate changes in the x_j's caused by changes in the b_i's. The jth component of Equation 3-18 is used.

$$x_j = \sum_{i=1}^{m} \beta_{ji} b_i \quad \text{for} \quad j = 1,2,\ldots,m \tag{3-24}$$

For a change in b_i of an amount Δb_i, the new value of x_j^*, called $x^*_{j,\text{new}}$ is:

$$x_{j,new} = \sum_{i=1}^{m} \beta_{ji}(b_i + \Delta b_i) \quad \text{for} \quad j = 1,2,\ldots,m$$

and

$$x_{j,new} = x^*_j + \sum_{i=1}^{m} \beta_{ji}\Delta b_i \quad \text{for} \quad j = 1,2,\ldots,m \tag{3-25}$$

For the optimal solution $\mathbf{x}^*$ to remain optimal the values of $x_{j,\text{new}}$ must not become negative. The problem must be resolved if any of the $x_{j,new}$'s becomes negative.

The change in the value of the objective function for changes in the b_i's, is computed using Equation (3-19). Since the left-hand side of Equation 3-19 is zero at the optimum, it can be written as:

$$p^* = -\sum_{i=1}^{m} b_i \lambda_i \tag{3-26}$$

Using the same procedure for the change Δb_i, the change in the value of the objective function is:

$$p^*_{new} = -\sum_{i=1}^{m} (b_i + \Delta b_i)\lambda_i$$

$$p^*_{new} = p^* - \sum_{i=1}^{m} \Delta b_i \lambda_i \tag{3-27}$$

It is from this equation that the Lagrange multipliers receive the name *shadow prices* since they have dimensions of dollars per unit and are used to compute the new value of the objective function from changes in the b_i's. This is called a *marginal cost calculation*.

Generally, in large linear programming computer programs part of the computations includes the calculation of $x^*_{j,\text{new}}$ and p^*_{new} for upper and lower limits on the b_i's. Also, values of the Δb_i's can be computed that will give the largest possible change in the x_j^*'s, i.e. $x_{j,\text{new}} = 0$. Simultaneous changes in the right-hand side of the constraint equations can be performed using the 100% rule, and this procedure is described by Bradley et al (19).

Example 4-7

For the problem given in Example 4-6, find the new optimal solution for $\Delta b_1 = -5$ without resolving the problem. Using Equation (3-25) to compute the changes in the x_i's gives:

$$x_{1,\text{new}} = x_1 + \beta_{11}\,\Delta b_1 + \beta_{12}\,\Delta b_2$$

$$x_{2,\text{new}} = x_2 + \beta_{21}\,\Delta b_1 + \beta_{22}\,\Delta b_2$$

Substituting in the values for $\Delta b_1 = -5$ and $\Delta b_2 = 0$ gives

$$x_{1,new} = 7\tfrac{1}{2} + 5/4(-5) \quad = 5/4$$

$$x_{2,new} = 2\tfrac{1}{2} + (-\tfrac{1}{4})\,(-5) = 15/4$$

Using Equation (4-27) the change in the objective function is computed as:

$$p^*_{new} = p^* - [\lambda_1\, \Delta b_1 + \lambda_2\, \Delta b_2] = 17\tfrac{1}{2} - (-9/4)\,(-5)$$

$$p^*_{new} = 25/4 = 6\tfrac{1}{4}$$

The optimal solution remains optimal, but the profit decreases from 17 ½ to 6¼.

Changes in the right-hand side of the constraint equations are part of the sensitivity analysis of the MPSX program. In Table 3-12(a) the smallest and largest values of the right-hand side of the constraint equations are given for the optimal solution to remain optimal as LOWER LIMIT and UPPER LIMIT. Also, the Lagrange multipliers were computed, and these are called the DUAL ACTIVITY in the MPSX nomenclature of Table 3-12(a). In this table NONE indicates that there is no bound, and a dot indicates that the value was zero. Correspondingly, in Table 3-12(b) the upper and lower limits on the variables are given. In this case the dot indicates that the lower bound was zero, and NONE indicates that there was no upper bound on the variable because BOUNDS was not used.

Changes in the Coefficients of the Objective Function: It is necessary to consider the effect on the optimal solution of changes in the cost coefficients of the variables in the basis and those not in the basis also. Referring to Equation (3-19), the coefficients of the variables that are not in the basis, i.e., $x_{m+1}, ..., x_n$ must remain negative for maximization.

$$\left[c_j + \sum_{i=1}^{m} a_{ij}\lambda_i\right] < 0 \quad \text{for} \quad j = m+1\ldots,n \tag{3-28}$$

If a coefficient becomes positive from a change in the cost coefficients, it would be profitable to have that variable enter the basis.

The values of the Lagrange multipliers are affected by changes in the cost coefficients of the variables in the basis, since they are related by Equation (3-23). The term in the brackets in Equation (3-28) is named the reduced cost (19), and it is convenient to define this term as c'_j to obtain the equation that accounts for the effect of changes in cost coefficients on the optimal solution.

$$c'_j = \left[c_j + \sum_{i=1}^{m} a_{ij}\lambda_i\right] < 0 \quad \text{for} \quad j = m+1\ldots,n \tag{3-29}$$

where c'_j must remain negative for the optimal solution to remain optimal for maximizing.

The Lagrange multipliers, λ_i's, are eliminated from Equation (3-29) by substituting Equation (4-23) to give:

$$c'_j = c_j - \sum_{i=1}^{m} a_{ij} \sum_{k=1}^{m} \beta_{ki} c_k \quad \text{for j=m+1,...,n}$$

or

$$c'_j = c_j - \sum_{i=1}^{m} \sum_{k=1}^{m} a_{ij} \beta_{ki} c_k \quad \text{for j=m+1,...,n} \tag{3-30}$$

For a change, Δc_j, in the non-basic variable cost coefficient, c_j, and for a change, Δc_k, in the basic variables cost coefficient c_k, it can be shown that the following equation holds:

$$c'_{j,new} = c' + \Delta c_j - \sum_{k=1}^{m} \Delta c_k \sum_{k=1}^{m} a_{ij} \beta_{ki} \quad \text{for j=m+1,...,n} \tag{3-31}$$

When maximizing, the new coefficients must remain negative for the variables not in the basis to have the optimal solution remain optimal, i.e.

$$c'_{j\ \text{new}} < 0 \tag{3-32}$$

If Equation (3-32) does not hold then a new optimal solution must be obtained by solving the linear programming problem with the new values of the cost coefficients.

If the optimal solution remains optimal, the new value of the objective function can be computed with the following equation:

$$p^*_{new} = p^* + \sum_{k=1}^{m} x_k \Delta c_k \tag{3-33}$$

If the problem must be resolved, it is usually convenient to introduce an artificial variable and proceed from this point to the new optimal solution. Large linear programming codes usually have this provision. Also, they can calculate a range of values of the cost coefficients where the optimal solution remains optimal and the corresponding effect on the objective function. The procedure used is called the 100% rule and is described by Bradley, et al. (19).

Example 3-8

For the problem given in Example 3-6 compute the effect of changing the cost coefficient c_1 from 2 to 3 and c_3 from 1 to 4, i.e. $\mathbf{\Delta}c_1 = 1$ and $\mathbf{\Delta}c_3 = 3$. Using Equation 3-31 produces the following results for $j = 3, 4, 5$ (since $\mathbf{\Delta}c_2 = 0$).

$$c'_{3,new} = c'_3 + \Delta c_1[a_{13}\beta_{11} + a_{23}\beta_{12}]$$

substituting

$$c'_{3,new} = -1 + 3 - (1)[(1)(5/4) + (1)(-1/4)] = 1$$

$$c'_{4,new} = c'_4 + \Delta c_4 - \Delta c_1[a_{14}\beta_{11} + a_{24}\beta_{12}]$$

substituting

$$c'_{4,new} = -9/4 + 0 - (1)[(1)(5/4) + (0)(-1/4)] = -13/4$$

$$c'_{5,new} = c'_5 + \Delta c_5 - \Delta c_1[a_{15}\beta_{11} + a_{25}\beta_{12}]$$

substituting

$$c'_{5,new} = -1/4 + 0 - (1)[(0)(5/4) + (-1)(-1/4)] = -1/2$$

An improvement in the objective function can be obtained, for $c'_{3,new}$ is greater than zero. Increasing x_3 from zero to a positive number will increase the value of the objective function. However, the problem will have to be resolved.

In the MPSX program, the RANGE command and the parametrics are used to find the range over which the variables, right-hand-sides and the coefficients of the objective function and constraints, may be varied without changing the basis for the optimal solution. Output from the RANGE command consists of four sections: sections 1 and 2 for rows and columns at their limit levels, and sections 3 and 4 for rows and columns at an intermediate level (in the basis) which will be described here. Further information is given in references (12, 15 and 16).

In Table 3-13 the RANGE output is shown for constraint rows at upper and lower limit levels. The first four columns have the same meaning as in the output from SOLUTION. The next four have two entries for each row. LOWER ACTIVITY and UPPER ACTIVITY are the lower and upper bounds on the range of values that the row activity (right-hand side) may have. Since the slack variable for the row is zero at a limit level, the upper and lower activities are numerically equal to the bounds of the range that the right-hand sides may have. The two UNIT COST entries are the changes in the objective function per unit change of activity when moving from the solution activity to either the upper or lower bound. The column labeled LIMITING PROCESS contains the name of the row or column that will leave the basis if the activity bounds are violated. The status column, AT, indicates the status of the leaving row or column. For example, in line 15 of Table 3-13 the row FOMIN is at its lower limit, its activity value is 10,000, and the right-hand side may take on values between 5,652.8 and 12,252.2 without changing the basis. If FOMIN exceeds 12,252.2, then SRFODF would leave the basis. If FOMIN goes below 5,652.8, then CCFODF would leave the basis. The cost associated with a change in FOMIN is \$27.18/bbl with profit decreasing for an increase in FOMIN.

Table 3·13. MPS Output, RANGE: Rows at Limit Level
Section 1 - rows at limit level

Number	Row	AT	Activity	Lower Activity Upper Activity	Unit Cost Unit Cost	Limiting Process	AT AT
4	PGBLEND	EQ	.	-1530.74	19.320	SRGPG	LL
				807.77	-19.320	RGMIN	LL
5	PGOCANE	LL	.	-75122 38	0.28	RGMIN	LL
				142358.38	-0.28	SRGPG	LL
8	RGBLEND	EQ	.	-157.39	19.320	CCGRG	LL
				184.70	-19.320	RFGRG	LL
9	RGOCTAN	LL	.	-18739.35	0.280	RFGRG	LL
				17326.68	-0.280	CCGRG	LL
10	RGVAPP	UL	.	-16460.63	.	RFGRG	LL
				9533.63	.	CCGRG	LL
12	DFBLEND	EQ	.	-4091. 76	40.320	CCFODF	LL
				541.56	-40.320	DFDENS	UL
14	DFSULFUR	UL	.	-331.08	.	SRFODF	LI.
				2045.89	.	CCFODF	LL
15	FOMIN	LL	10000.0	5652.8	27.180	SRFODF	LL
				12252.2	-27.180	SRFODF	LL
16	FOBLEND	EQ	.	-4347.24	40.320	CCFOFO	LL
				1941.99	-40.320	FODENS	UL
19	ADCAP	UL	100000.0	94572.99	-8.154	DFMIN	LL
				105485.23	8.154	RFCAP	UL
20	ADFGYLD	EQ	.	-INFINITY	0.01965	NONE	
				3541999.0	-.01965	FGAD	LL
21	ADSRGYLD	EQ	.	-26197.55	41.300	PGMIN	UL
				5180.85	-41.300	RGMIN	LL
22	ADNYLD	EQ	.	-1300.0	45.570	RFCAP	LL
				13394.25	-45.570	RFGPG	LL
23	ADDSYLD	EQ	.	-12733.73	40.320	SRFODF	LL
				2490.99	-40.320	DFMIN	LL
24	ADFOYLD	EQ	.	-4347.24	40.320	CCFOFO	LL
			.	2252.22	-40.320	SRFODF	LL
26	FGRYLD	EQ	.	-INFINITY	0.01965	NONE	
				3761190.0	-.01965	FGRF	LL
27	RFRFGYLD	EQ	.	-6829.31	48.440	RGMIN	LL
				12429.87	-48.440	RFGFG	LL
28	CCAP	UL	30000,00	25926.81	-5.274	CCFOFO	LL
				32886.36	5.274	SRFODF	LL
29	CCFGYLD	EQ	.	-INFINITY	0.01965	NONE	
				11591992.0	-.01965	FGCCF	LL

Table 3·13. Continued

Number	Row	AT	Activity	Lower Activity Upper Activity	Unit Cost Unit Cost	Limiting Process	AT AT
30	CCGYLD	EQ	.	-107317 .69	45.556	RGMIN	LL
				15646.77	-45.556	CCGPG	LL
31	CCFGYL	EQ		-28457.97	40.320	SRFDFO	LL
				2252.22 -	40.320	SRFODF	LL
32	SRGSPLIT	EQ	.	-26197.55	`41.300	PGIMIN	LL
				5180.85	-41.300	RGMIN	LL
33	SRNSPLIT	EQ	.	-1300.0	45.570	RFCAP	UL
				13394.25	-45.570	RFGFG	LL
34	SRDSSPLT	EQ	.	12733.73	40.320	SRFJDF	LL
				2490.9	-40.320	DFMIN	LL
35	SRFOSPLT	EQ	.	-4347.24	40.320	CCFOFO	LL
				2252.22	-40.320	SRFODF	LL
36	RFGSPLIT	EQ	.	-6829.87	48.440	RGIIN	LL
				12429.87	-48.440	RFGFG	LL
37	CCGSPLIT	EQ	.	-107317.69	45.566	RGMIN	LL
				15646.77	-45.566	CCGPG	LL
38 CCFOSPLT		EQ	.	-28457.97	40.320	SRFDFO	LL
				2252.22	-40.320	SREODF	LL

Similar information is provided in Table 3-14 about the range over which the nonbasis activities (variables) at upper or lower limits may be varied without forcing the row or column in LIMITING PROCESS out of the basis. An additional column is included in the table, LOWER COST/UPPER COST to show the highest and lowest cost coefficients at which the variable will remain in the basis. If the objective function cost coefficient goes to the LOWER COST, the activity will increase to UPPER ACTIVITY. Similarly, if its cost goes below UPPER COST, the activity will be decreased to LOWER ACTIVITY.

The third section of output from the range study is given in Table 3-15. It contains information about constraints that are not at their limits and, therefore, are in the basis of the optimal solution. The column headings have the same meaning as the headings for section 1 except that here the variable listed under LIMITING PROCESS will enter the basis if the bounds are exceeded.

The fourth section, shown in Table 3-16, gives the RANGE analysis of the variables listed under the columns in the basis. As in Table 3-15 the variable listed under LIMITING PROCESS will enter the basis when activity is forced beyond the upper or lower activity bounds.

Table 3-14_ MPS Output, RANGE: Columns at Limit Level

SECTION 2 – Columns at Limit Level

Number	Column	AT	Input Cost	Lower Activity / Upper Activity	Unit Cost / Unit Cost	Lower Cost / Upper Cost	Limiting Process	AT / AT
48	SRDOCC	LL	-2.20	-1964.99	5.353	-Infinity	SRFODF	LL
				4550.96	-5.353	3.128	CCFOFO	LL
55	SRNPG	LL	.	-1300.00	8.051	-Infinity	RFCAP	UL
				3725.88	-8.051	8.046	SRGPG	LL
60	SRNRG	LL	.	-615.85	8.051	-Infinity	RFGRG	LL
				543.39	-8.051	8.046	CCGRG	LL
63	SRNOF	LL	.	-1300.00	5.250	-Infinity	RFCAP	UL
				9428.02	-5.250	5.251	CCFODF	LL
69	SRDSFO	LL	.	-1913.74	.	-Infinity	SRFODF	UL
				4596.20	.	0.000	CCFOFO	LL

Table 3-15 MPS Output, RANGE: Rows at Intermediate Level

Section 3 - rows at intermediate level

Number	Row	AT	Activity	Slack Activity	Lower Activity / Upper Activity	Unit Cost / Unit Cost	Limiting Process	AT / AT
2	CRDAVAIL	BS	100000.0	10000.0	94572.98	-8.154	ADCAP	UL
					100000.0	-INFINITY	NONE	
3	PGMIN	BS	47113.0	-37113.2	23655.1	-1.278	SRNPG	LL
					47113.2	-INFINITY	NONE	
6	PGVAPP	BS	-188607.2	188607.21	-188607.16	-INFINITY	NONE	
					-172146.52	.	RGVAPP	UL
7	RGMIN	BS	22520.4	-12520.4	21167.53	-32.710	ADCAP	LL
					46246.79	-1.264	SRNPG	UL
11	DFMIN	BS	12490.9	-2490.9	9592.13	-17.765	ADCAP	UL
					21919.02	-5.251	SRNDF	LL
3	DFDENS	BS	-165458.8	165458.8	-165775.57	.	DFSULFUR	UL
					-153666.96	.	SRDSFO	LL
17	FODENS	BS	-571996.8	571966.8	-583788.53	.	SRDSFO	LL
					-571679.93	.	DFSULFUR	UL
18	FOSULFUR	BS	-22286.7	22286.7	-27917.23	-10.872	FOMIN	LL
					-21955.59	.	DFSULFUR	UL
25	RFCAP	BS	23700.0	1300.0	14271.68	-5.251	SRNDF	UL
					23700.00	-INFINITY	NONE	

The information of greatest interest here are the entries for columns with coefficients in the objective function. These are: CRUDE (39), FGAD(40), SRNRF(45), FGRF(46), SRFOCC(49), FCCC(50), PG(57), RG(62), DF(67), and FO(71). Examining the first row in Table 3-16, one finds that if the cost coefficient becomes -41.15, the activity (crude flow rate) would be reduced from 100,000 to 94,572.98. Consequently, if the cost of crude oil is increased to \$40.09/bbl (operating cost is \$1.00/bbl) the refinery should reduce its throughput by only 5.2%. Also notice that the lower cost for premium gasoline (PG) is 44.082 while the input cost is 45.35. If the bulk sale price of premium gasoline were to drop to \$44.08/bbl., it would be profitable for the refinery to produce 23,661 bbl/day, a drop of almost 50% from the optimum value of 47,111bbl/day currently produced. A similar analysis for fuel oil (FO) indicates that it will probably never be profitable to produce fuel oil since the sale price would have to increase from \$13.14/bbl to \$40.32/bbl before production should be increased above the minimum.

Changes in Coefficients of the Constraint Equations: Referring to Equation 3-29 it is seen that changes in the a_{ij}'s for the non-basic variables will cause changes in c'_j. For the optimal solution to remain optimal $c'_j < 0$ when maximizing; and if not, the problem must be resolved. To evaluate the changes in the coefficients of the constraint equations, a_{ij}, several pages of algebraic manipulations are required. This development is similar to the ones given here for the b_i's and c_j's, and is discussed in detail by Garvin (3) and Gass (4) along with the subject of parametric programming, i.e., evaluating a set of ranges on the a_{ij}'s, b_i's and c_j's where the optimal solution remains optimal. Due to space limitations these results will not be given here. Also, the MPSX code has the capability of making these evaluations as previously mentioned.

Addition of New Variables: The effect of adding new variables can be determined by modifying Equation 4-19. If k new variables are added to the problem then k additional terms will be added to Equation 4-19, and the coefficient of the kth term is:

$$\left[c_{n+k} + \sum_{i=1}^{m} a_{i,n+k}\lambda_i\right] \tag{3-34}$$

Each of these k terms can be computed with the available information. If all of these are less than zero, the original optimal solution remains at the maximum. If Equation 3-34 is greater than zero, the solution can be improved; and the problem has to be resolved. Artificial variables are normally used to evaluate additional variables to obtain new optimal solution.

Table 3-16 MPS Output, RANGE: Columns at Intermediate Levels

Section 4 – columns at intermediate level

Number	Row	AT	Activity	Input Cost	Lower Activity Upper Activity	Unit Cost	Lower Cost Upper Cost	Limiting Process	AT
39	CRUDE	BS	100000.0	-33.0	94573.0	-8.154	-41.154	ADCAP	UL
					100000.0	-INFINITY	INFINITY	NONE	
40	FGAD	BS	3541999.0	0.01965	3349774.0	-.2302	-0.210	ADCAP	UL
					3541999.0	-INFINITY	INFINITY	NONE	
41	SRG	BS	27000.0	.	25534.7	-30.200	-30.201	ADCAP	UL
					27000.0	-INFINITY	INFINITY	NONE	
42	SRN	BS	23699.9	.	22413.8	-34.405	-34.405	ADCAP	UL
					23699.9	-INFINITY	INFINITY	NONE	
43	SRDS	BS	8699.9	.	8227.8	-93.726	-93.726	ADCAP	UL
					8699.9	-INFINITY	INFINITY	NONE	
44	SRFO	BS	37199.9	.	35181.1	-21.919	-21.919	ADCAP	UL
					37199.9	-INFINITY	INFINITY	NONE	
45	SRNRF	BS	23699.9	-2.50	14271.9	-5.251	-7.750	SRNDF	UL
					23699.9	-INFINITY	INFINITY	NONE	
46	FGRF	BS	3761190.0	0.01965	2264964.1	-.0331	-.0134	SRNDF	LL
					3761190.	–INFINITY	INFINITY	NONE	
47	RFG	BS	21993.6	.	13244.4	-5.658	-5.658	SRNDF	UL
					21993.6	-INFINITY	INFINITY	NONE	
49	SRFOCC	BS	30000.0	-2.20	25926.8	-5.274	-7.474	CCCAP	UL
					30000.0	-INFINITY	INFINITY	NONE	
50	FGCC	BS	11591992.0	0.01965	10018114.0	-.01365	0.006	CCCAP	UL
					11591992.0	-INFINITY	INFINITY	NONE	
51	CCG	BS	20640.0	.	17837.6	-7.665	-7.665	CCCAP	UL
					20640.0	-INFINITY	INFINITY	NONE	
52	CCFO	BS	6590.9	.	5696.1	-24.003	-24.003	CCCAP	UL
					6591.0	-INFINITY	INFINITY	NONE	
53	SRGPG	BS	13852.0	.	10510.6	-1.309	-1.309	SRNRG	LL
					17073.2	.	.	RGVAPP	UL
54	RFGPG	BS	17240.0	.	12541.4	-0.931	-0.931	SRNRG	LL
					21993.6	.	.	RGVAPP	UL
56	CCGPG	BS	16021.2	.	8046.4	.	.	RGVAPP	UL
					20640.0	-0.947	0.947	SRNRG	LL
57	PG	BS	47113.2	45.36	23655.1	-1.279	44.081	SRNPG	LL
					47113.2	-INFINITY	INFINITY	NONE	
58	SRGRG	BS	13148.0	.	9926.8	.	.	RGVAPP	UL
					16489.4	-1.309	1.309	SRNRG	LL
59	RFGRG	BS	4753.6	.	-4796.2	.	.	RGVAPP	UL
					8947.9	-1.043	1.043	SRNRG	LL

Table 3-16 MPS Output, RANGE: Columns at Intermediate Levels

Section 4 – columns at intermediate level

Number	Row	AT	Activity	Input Cost	Lower Activity / Upper Activity	Unit Cost	Lower Cost / Upper Cost	Limiting Process	AT
61	CCGRG	BS	4618.8	.	-12328.6	-0.947	0.947	SRNRG	LL
					12593.6	.	.	RGVAPP	UL
62	RG	BS	22520.4	43.68	21167.5	-32.710	10.970	ADCAP	LL
					46246.8	-1.264	44.944	SRNPG	LL
64	CCFODF	BS	3263.0	.	-1372.7	-15.172	15.172	SRNDF	LL
					3791.0	.	-0.000	DFSULFUR	UL
65	SRDSDF	BS	8700.0	.	4103.8	.	0.000	SRDSFO	LL
					8700.0	-INFINITY	INFINITY		NONE
66	SRFODF	BS	528.0	.	-2800.0	.	.	DFSULFUR	UL
					1796.2	.	-0.000	SRDSFO	LL
67	DF	BS	12491.0	40.32	10000.0	-17.7652	2.555	ADCAP	UL
					21919.0	-5.250	45.570	SRNDF	LL
68	CCFOFO	BS	3328.0	.	2800.0	.	0.000	DFSULFUR	UL
					6591.0	5.172	15.172	SRNDF	LL
70	SRFOFO	BS	6672.0	.	5403.8	.	.	SRDSFO	LL
					7200.0	.	.	DFSULFUR	UL
71	FO	BS	10000.0	13.14	10000.0	-INFINITY	-INFINITY		NONE
					12252.2	-27.180	40.320	FOMIN	LL

Addition of More Constraint Equations: For the addition of more constraint equations the procedure is to add artificial variables and proceed with the solution to the optimum. The artificial variables supply the canonical form for the solution. The following example shows the effect of adding an additional independent variable and an additional constraint equation to a linear programming problem to illustrate the application of the methods described above.

Example 3-9

Solve the linear programming problem using the Simplex Method

minimize: $x_1 - 3x_2$

subject to: $3x_1 - x_2 \leq 7$

$-2x_1 + 4x_2 \leq 12$

Introduce slack variables x_3 and x_4 for an initially feasible basis and ignore the terms with x_5 in parentheses for now. This gives:

$$
\begin{array}{llll}
x_1 - 3x_2 & (+\,2x_5) & = c & c = 0 \\
3x_1 - x_2 + x_3 & (+\,2x_5) & = 7 & x_3 = 7 \\
-2x_1 + 4x_2 \quad + x_4 & & = 12 & x_4 = 12 \\
 & & & x_1 = 0 \\
 & & & x_2 = 0
\end{array}
$$

Applying the Simplex Method x_2 enters and x_4 leaves the basis. Performing the algebraic manipulations gives:

$$
\begin{array}{llll}
-\,0.5x_1 \qquad\quad +\ 0.75x_4 & (+\,2x_5) & = c + 9 & c = -9 \\
2.5x_1 \quad + x_3 + 0.25x_4 & (+\,2x_5) & = 10 & x_3 = 10 \\
-\,0.5x_1 + x_2 \quad + 0.25x_4 & & = 3 & x_2 = 3 \\
 & & & x_1 = 0 \\
 & & & x_4 = 0
\end{array}
$$

Applying the Simplex Method x_1 enters and x_3 leaves the basis giving the following results:

$$
\begin{array}{lll}
0.2x_3 + 0.8x_4\ (+\,2.4x_5) = c + 11 & & c = -11 \\
x_1 + \quad 0.4x_3 + 0.1x_4\ (-\,0.8x_5) = 4 & & x_1 = 4 \\
x_2 + \quad 0.2x_3 + 0.3x_4\ (+\,0.4x_5) = 5 & & x_2 = 5 \\
 & & x_3 = 0 \\
 & & x_4 = 0
\end{array}
$$

The optimal solution has been obtained since all of the coefficients of the variables in the objective function (not in the basis) are positive.

We compute the inverse of the optimal basis **A*-**1 and the Lagrange multipliers, having obtained the optimal solution as follows:

$$A^* = \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} \quad |A^*| = 10 \quad \|A_{ij}^*\| = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \quad A^{*-1} = \begin{bmatrix} 2/5 & 1/10 \\ 1/5 & 3/10 \end{bmatrix}$$

For Lagrange multipliers Equation 3-22 is used:

$$\boldsymbol{\lambda} = -[\mathbf{A}^{*-1}]^T \mathbf{c}$$

and substituting gives

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = -\begin{bmatrix} 2/5 & 1/10 \\ 1/5 & 3/10 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix}$$

If the first constraint equation is changed as follows by adding another variable x_5:

$$3x_1 - x_2 + 2x_5 \leq 7$$

and the objective function is changed by including x_5 as shown below:

$$x_1 - 3x_2 + 2x_5$$

Determine how this addition of a new variable affects the optimal solution found previously. The linear programming problem now has the following form:

$$\begin{aligned} x_1 - 3x_2 + 2x_5 \qquad &= c \\ 3x_1 - x_2 + 2x_5 + x_3 \quad &= 7 \\ -2x_1 + 4x_2 \qquad + x_4 &= 12 \end{aligned}$$

To determine if the optimal solution remains optimal, Equation 3-34 is used. For this problem $n = 4$, $k = 1$ and $m = 2$, and Equation 3-34 has the form:

$$[c_5 + a_{1,5}\lambda_1 + a_{2,5}\lambda_2]$$

substituting gives:

$$[2 + 2(1/5) + 0(4/5)] = 2.4 > 0$$

The optimal solution remains optimal since Equation 3-34 is positive for this case, and it is not necessary to resolve the problem. x_5 is not in the basis and has a value of zero.

The terms in parenthesis show the solution with the additional variable included. As can be seen the coefficient at the final step is the same as computed using Equation 3-34.

Find the new optimal solution if the following constraint equation is added to the problem

$$-4x_1 + 3x_2 + 8x_5 + x_6 = 10$$

The constraint equation is added to the optimal solution set so the problem will not have to be completely solved and is:

$$\begin{aligned}
&0.2x_3 + 0.8x_4 + 2.4x_5 &&= c + 11\\
x_1 + \quad &0.4x_3 + 0.1x_4 - 0.8x_5 &&= 4\\
x_2 + \quad &0.2x_3 + 0.3x_4 + 0.4x_5 &&= 5\\
-4x_1 + 3x_2 \quad &+ 8x_5 + x_6 &&= 10
\end{aligned}$$

x_6 is used as the variable in the basis from the additional constraint equation. x_1 and x_2 are eliminated from the added constraint equation by algebraic manipulation and gives:

$$\begin{aligned}
&0.2x_3 + 0.8x_4 + 2.4x_5 &&= c + 11 &&c = -11\\
x_1 \quad &+ 0.4x_3 + 0.1x_4 - 0.8x_5 &&= 4 &&x_1 = 4\\
x_2 \quad &+ 0.2x_3 + 0.3x_4 + 0.4x_5 &&= 5 &&x_2 = 5\\
&x_3 - 0.5x_4 + 10x_5 + x_6 &&= 11 &&x_6 = 11\\
& && &&x_4 = 0\\
& && &&x_5 = 0
\end{aligned}$$

The new optimal solution has been found since all of the coefficients in the objective function are positive. Artificial variables would normally have been used, especially in a computer program, to give a feasible basis and proceed to the optimum.

Closure

In this chapter the study of linear programming was taken through the use of large computer codes to solve industrial problems. Sufficient background was provided to be able to formulate and solve linear programming problems for an industrial plant using one of the large linear programming codes and to interpret the optimal solution and associated sensitivity analysis. In addition, this background should provide the ability for independent reading on extensions of the subject.

The mathematical structure of the linear programming problem was introduced by solving a simple problem graphically. The solution was found to be at the intersection of constraint equations. The Simplex Algorithm was then presented which showed the procedure of moving from one intersection of constraint equations (basic feasible solution) to another and having the objective function improve at each step until the optimum was reached. Having seen the Simplex Method in operation, the important theorems of linear programming were discussed which guaranteed that the global optimum would be found for the linear objective function and linear constraints. Then methods were presented which illustrated how a process flow diagram and associated information could be converted to a linear programming problem to optimize an industrial process. This was illustrated with a simple petroleum refinery example, and the solution was obtained using a large standard linear programming code, Mathematical Programming System Extended (MPSX), on an IBM 4341 computer. The chapter was included with a discussion of post-optimal analysis procedures that evaluated the sensitivity of the solution to changes in important parameters of linear programming problem. This sensitivity analysis was illustrated using simple examples and results from the solution of the simple refinery using the MPSX code.

A list of selected references is given at the end of the chapter for information beyond that presented here. These texts include the following topics. The Revised Simplex Method is a modification of the Simplex Method that permits a more accurate and rapid solution using digital computers. The dual linear programming problem converts the original or primal problem into a corresponding dual problem that may be solved more readily than the original problem. Parametric programming is an extension of sensitivity analysis where ranges on the parameters, a_{ij}'s, b_i's and c_j's, are computed directly considering more than one parameter at a time. Also, there are decomposition methods that take extremely large problems and separate or decomposes them into a series of smaller problems that can be solved with reasonable computer time and space. In addition, special techniques have been developed for a class of transportation and network problems that facilitate their solution. Linear programming has been extended to consider multiple conflicting criteria, i.e., more than one objective function, and this has been named *goal programming*. An important extension of linear programming is the case where the variables can take on only integer values, and this has been named *integer programming*. Moreover, linear programming and the theory of games have been interfaced to develop optimal strategies. Finally, almost all large computers have one or more advanced linear programming codes capable of solving problems with thousands of constraints and thousands of variables. It is very time consuming and tedious task to assemble and enter reliable data correctly in using these programs. These codes, e.g. MPSX, are very efficient and use sparse matrix inversion techniques, methods for dealing with ill-conditioned matrices, structural data formats and simplified input and output transformations. Also, they usually incorporate post optimal ranging, generalized upper bounding and parametric programming (9,12). Again, the topics mentioned above are discussed in the articles and books in the References and the Selected List of Texts at the end of the chapter.

Selected List of Texts on Linear Programming and Extensions

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Problems

3-1. Solve the following problem by the Simplex Method:

$$\text{Maximize:} \quad 6x_1 + x_2 = p$$

$$\text{Subject to:} \quad 3x_1 + 5x_2 \leq 13$$

$$6x_1 + x_2 \leq 12$$

$$x_1 + 5x_2 \leq 10$$

Determine the range on x_1 and x_2 for which the optimal solution remains optimal. Explain. (Note: It is not necessary to use sensitivity analysis.)

3-2. Solve the following problem by the Simplex Method:

$$\text{Maximize:} \quad x_1 + 2x_2 + 3x_3 - x_4 = p$$

$$\text{Subject to:} \quad x_1 + 2x_2 + 3x_3 + x_5 = 15$$

$$2x_1 + x_2 + 5x_3 + x_6 = 20$$

$$x_1 + 2x_2 + x_3 + x_4 = 10$$

Start with x_4, x_5, and x_6 in the basis.

3-3. a. Solve the following problem by the Simplex Method:

$$\text{Maximize:} \quad 2x_1 + x_2 = p$$

$$\text{Subject to:} \quad x_1 + x_2 \leq 6$$

$$x_1 - x_2 \leq 2$$

$$x_1 + 2x_2 \leq 10$$

$$x_1 - 2x_2 \leq 1$$

b. Compute the inverse of the optimal basis and the largest changes in b_i's for the optimal solution remain optimal.

3-4. Solve the following problem by the Simplex Method:

Maximize: $3x_1 + 2x_2 = p$

Subject to: $x_1 + x_2 \leq 8$

$2x_1 + x_2 \leq 10$

3-5. a. Solve the following problem by the Simplex Method:

Maximize: $x_1 + 2x_2 = p$

Subject to: $x_1 + 3x_2 \leq 105$

$-x_1 + x_2 \leq 15$

$2x_1 + 3x_2 \leq 135$

$-3x_1 + 2x_2 \leq 15$

b. Solve this problem by the classical theory using Lagrange multipliers, and explain why Lagrange multipliers are sometimes called "shadow" or "implicit" prices.

3-6. a. Solve the following problem by the Simplex Method using slack and artificial variables:

Maximize: $x_1 + 10x_2 = p$

Subject to: $-x_1 + x_2 \geq 5$

$3x_1 + x_2 \leq 15$

b. Calculate the inverse of the optimal basis and the Lagrange multipliers.

c. Calculate the largest changes in the right-hand side of the constraint equations (b_j's) for the optimal solution in part a to remain optimal.

3-7. Solve the following problem by the Simplex Method using the minimum number of slack, surplus, and artificial variables needed for an initially feasible basis.

Minimize: $2x_1 + 4x_2 + x_3 = c$

Subject to: $x_1 + 2x_2 - x_3 \leq 5$

$2x_1 - x_2 + 2x_3 = 2$

$$-x_1 + 2x_2 + 2x_3 \geq 1$$

3-8. a. Solve the following problem using the Simplex Method using an artificial variable x_6 in the second constraint equation and adding the term $-10^6 x_6$ to the objective function.

$$\text{Maximize:} \quad 2x_1 + x_2 + x_3 = p$$

$$\text{Subject to:} \quad x_1 + x_2 + x_3 \leq 10$$

$$x_1 + 5x_2 + x_3 \geq 20$$

b. Compute the effect of changing cost coefficient c_1 from 2 to 3, i.e. $\Delta c_1 = 1$, and c_3 from 1 to 4, i.e., $\Delta c_3 = 3$ using the results of Example 4-6.

c. Without resolving the problem, find the new optimal solution if the first constraint equation is changed to the following by using the results of Example 4-6:

$$x_1 + x_2 + x_3 \leq 5$$

Also, compute the new optimal values of x_1 and x_2 and value of the objective function.

3-9. Consider the following linear programming problem:

$$\text{Maximize:} \quad 2x_1 + x_2 = p$$

$$\text{Subject to:} \quad x_1 + 2x_2 \leq 10$$

$$2x_1 + 3x_2 \leq 12$$

$$3x_1 + x_2 \leq 15$$

$$x_1 + x_2 \geq 4$$

a. Solve the problem by the Simplex Method using slack variables in the first three equations and an artificial variable in the fourth constraint equation as the initially feasible basis.

b. The following matrix is the inverse of the optimal basis, A^{*-1}. Multiply this matrix by the matrix A^* to obtain the unit matrix I:

$$A^{-1} = \begin{bmatrix} 0 & -0.143 & 0.429 & 0 \\ 0 & 0.429 & -0.286 & 0 \\ 1 & -0.714 & 0.143 & 0 \\ 0 & 0.286 & 0.143 & -1 \end{bmatrix}$$

c. Compute the Lagrange multipliers for the problem.
d. Compute the changes in the right-hand side of the constraint equations that will cause all of the values of the variables in the basis to become zero.

3-10.[3] Consider the following problem based on a blending analysis:

$$\text{Minimize:} \quad 50x_1 + 25x_2 = c$$

$$\text{Subject to:} \quad x_1 + 3x_2 \geq 8$$

$$3x_1 + 4x_2 \geq 19$$

$$3x_1 + x_2 \geq 7$$

a. Solve this problem by the Simplex Method.
b. Compute the inverse of the optimal basis and the Lagrange multipliers.
c. Determine the effect on the optimal solution (variables and cost) if the right-hand side of the second constraint equations is changed from 19 to 21 and the right-hand side of the third constraint equations is changed from 7 to 8.
d. Show that the following must hold for the optimal solution to remain optimal considering changes in the cost coefficients.

$$3/4 \leq c_1/c_2 \leq 3$$

3-11. Consider the following linear programming problem:

$$\text{Maximize:} \quad x_1 + 9x_2 + x_3 = p$$

$$\text{Subject to:} \quad x_1 + 2x_2 + 3x_3 \leq 9$$

$$3x_1 + 2x_2 + 2x_3 \leq 15$$

a. Solve this problem by the Simplex Method.
b. Compute the inverse of the optimal basis and the Lagrange multipliers.
c. Determine the largest changes in the right-hand side and in the cost coefficients of the variables in the basis for the optimal solution to remain optimal.

3-12. Solve the following problem by the Simplex Method. To demonstrate your understanding of the use of slack and artificial variables, use slack variables in the first two constraint equations and an artificial variable in the third constraint equation as the initially feasible basis:

$$\text{Maximize:} \quad x_1 + 2x_2 = p$$

Subject to: $-x_1 + x_2 \le 2$

$x_1 + x_2 \le 6$

$x_1 + x_2 \ge 1$

3-13. a. Derive Equation 4-31 from Equation 3-30. Explain the significance of the terms in Equation 3-31, and discuss the application of this equation in sensitivity analysis associated with coefficients of the variables in the objective function.

b. Starting with Equation 3-25 show that the change, in **b** which gives the limit on **Δb** for $x_{i,\ \text{new}}^{*} = 0$ is equal to -**b**.

3-14. In a power plant that is part of a chemical plant or refinery both electricity and process steam (high and low pressure) can be produced. A typical power plant has constraints associated with turbine capacity, steam pressure and amounts, and electrical demand. In Stoecker (14) the following economic and process model is developed for a simple power plant producing electricity, high pressure steam x_1, and low-pressure steam x_2.

$$\text{Maximize: } 0.16x_1 + 0.14x_2 = p$$

$$\text{Subject to:} \quad x_1 + x_2 \le 20$$

$$x_1 + 4x_2 \le 60$$

$$4x_1 + 3x_2 \le 72$$

Determine the optimal values of x_1 and x_2 and the maximum profit using the Simplex Method.

3-15. A company makes two levels of purity of a product that is sold in gallon containers. Product A is of higher purity than product B with profits of \$0.40 per gallon made on A and \$0.30 per gallon made on B. Product A requires twice the processing time of B, and if all B is produced, the company could make 1,000 gallons per day. However, the raw material supply is sufficient for only 800 gallons per day of both A and B combined. Product A requires a container of which only 400 1-gallon containers per day are available while there are 700 1-gallon containers per day available for B. Assuming the entire product can be sold of both A and B, what volumes of each should be produced to maximize the profit? Solve the problem graphically and by the Simplex Method.

3-16. A wax concentrating plant, as shown in Figure 3-9, receives feedstock with a low concentration of wax and refines it into a product with a high concentration of wax. In Stoecker (14) the selling prices of the products are x_1, \$8 per hundred pounds; and x_2, \$6 per hundred pounds; and the raw material costs are x_3, \$1.5 per hundred pounds, and x_4, \$3 per hundred pounds.

The plant operates under the following constraints:

a. The same amount of wax leaves the plant as enters it.

b. The receiving facilities of the plant are limited to no more than a total of 800 pounds per hour.

c. The packaging facilities can accommodate a maximum of 600 pounds per hour of x_2 and 500 pounds per hour of x_1.

If the operating cost of the plant is constant, use the Simplex Algorithm to determine the purchase and production plan that result in the maximum profit.

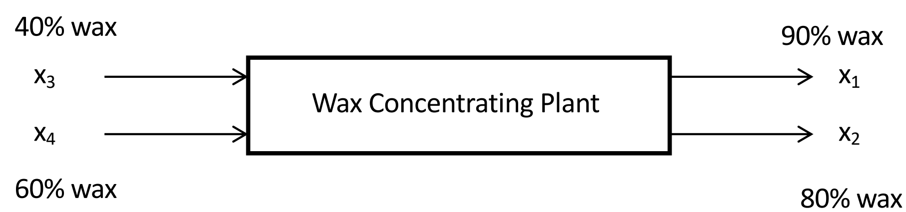

Figure 3-9 Wax Concentrating Plant for Problem 3-16

3-17. A company produces a product and a byproduct, and production is limited by two constraints. One is on the availability raw material, and the other is on the capacity of the processing equipment. The product requires 3.0 units of raw material and 2.0 units of processing capacity. The byproduct requires 4.0 units of raw materials and 5.0 units of processing capacity. There is a total of 1,700 units of raw material available and a total of 1600 units of processing capacity. The profit is $2.00 per unit for the product and $4.00 per unit for the by-product.

The economic model and constraints are:

Maximize: $2x_1 + 4x_2$

Subject to: $3x_1 + 4x_2 \leq 1700$ raw material constraint

$2x_1 + 5x_2 \leq 1600$ processing capacity constraint

a. Determine the maximum profit and the production of the product x_1 and byproduct x_2 using the Simplex Method.

b. Calculate the inverse of the optimal basis and the Lagrange multipliers.

c. i. If the total raw material available is increased from 1700 to 1701, determine the new product, byproduct and profit.

ii. If an additional 10 units of processing capacity can be obtained at a cost of $7, i.e. 1600 is increased to 1610, is this additional capacity worth obtaining?

d. A second by-product can be produced which requires 4.0 units of raw material and 3 1/3 units of processing capacity. Determine the profit that would have to be made on this by-product to consider its production.

3-18.[14] A chemical plant, whose flow diagram is shown in Figure 3-10, manufactures ammonia, hydrochloric acid, urea, ammonium carbonate, and ammonium chloride from carbon dioxide, nitrogen, hydrogen, and chlorine. The x values in Figure 3-10 indicate flow rates in moles per hour.

The costs of the feed stocks are c_1, c_2, c_3 and c_4; the values of the products are p_5, p_6, p_7 and p_8 in dollars per mole where the subscript corresponds to that of the x value. In reactor 3 the ratios of molar flow rates are $m = 3x_7$ and $x_1 = 2x_7$ and, in other reactors, straightforward material balances apply. The capacity of reactor 1 is equal to or less than 2,000 mol/hr of NH3 and the capacity of reactor 2 is equal to or less than 1,500 mol/hr of HCl as given by Stoecker (14).

a. Develop the expression for the profit.

b. Write the constraint equations for this plant.

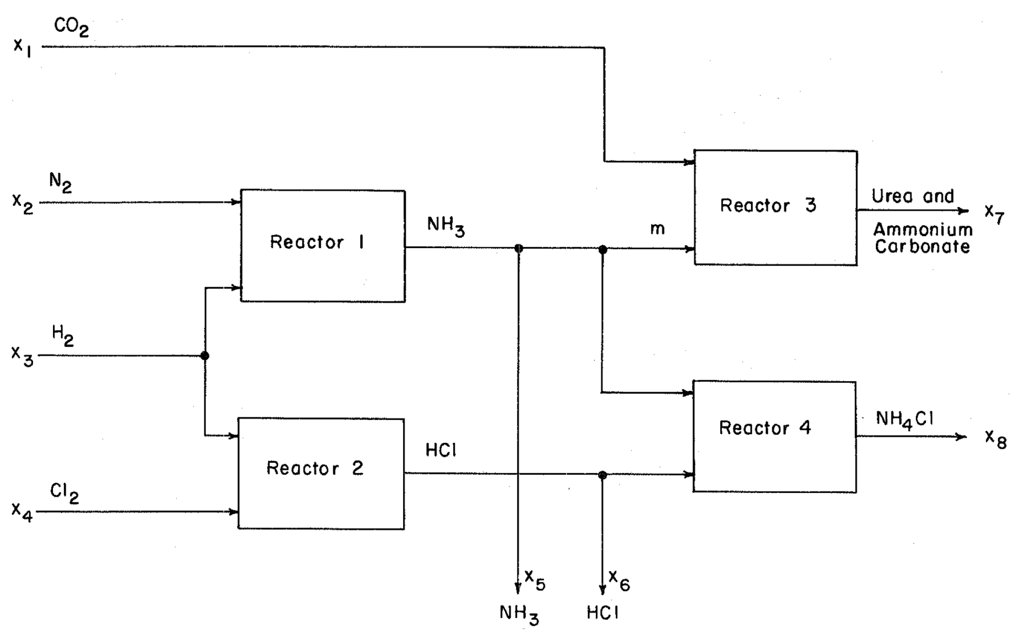

Figure 3.10 Flow Diagram of a Chemical Plant in Problem 4-18 (after Stoecker (14))

3-19.[8] The flow diagram of a simple petroleum refinery is shown in Figure 3-11. The prices and quality specifications of the products and their minimum production rates are given below:

Product	Quality	Minimum Production (bbl/day)	Prices($/bbl)
Premium Gasoline	$\geq$ 91 Mon	25,000	$ 45.00
Regular Gasoline	$\geq$ 89 Mon	10,000	43.50
Fuel Oil	< 55 Cont. No.	30,000	13.00

The current cost of crude is $32.00/barrel. Operating cost for separation in the crude still is $0.25 per barrel. for each product produced. The operating cost for the catalytic cracking unit is $0.10 for the straight run distillate and $0.15 for the straight run fuel oil.
The following table gives the specifications for each blending component:

Component	MON	Cont. No
Hv. Cat. Cycle Oil	-	59
Lt. Cat. Cycle Oil	88	50
Cat. Naphtha	97	-
Straight Run Distillate	84	-
Straight Run Gasoline	92	-

The capacity of the catalytic cracking unit must not exceed 50,000 barrels/day and the crude still is limited to 100,000 barrels/day. The crude is separated into three volume fractions in the crude still, 0.2 straight run gasoline, 0.5 straight run distillate, and 0.3 straight run fuel oil. In the catalytic cracking unit, a product distribution of 0.7 barrel of cat. naphtha, 0.4 light cat. cycle oil and 0.2 barrel of heavy cat. cycle oil is obtained per barrel of straight run distillate. The straight run fuel oil product distribution is 0.1 barrel of cat. naphtha, 0.3 barrel of light cat. cycle oil and 0.7 barrel of heavy cat. cycle oil.

Present a matrix representation of this simple refinery similar to the one shown in Figure 4-8. Be sure to include the objective function and material balance, unit, and blending constraints.

3-20. For the results of the MPSX optimization of the simple refinery consider the following:

a. In Table 3-12(b), it shows that the variable SRNPG is not in the basis. Compute the largest change in the cost coefficient of SRNPG for the optimal solution to remain optimal. Confirm that this is the correct answer by the sensitivity analysis results tabulated in the chapter.

b. In Table 3-12(b) the fuel oil (FO) flow rate is at the optimal value of 10,000 bbl/day. Compute the change in the profit if the fuel oil flow rate is increased to 11,000 bbl/day using Lagrange multipliers. Would this change cause the problem to be resolved according to the MPSX results, why?

c. The marketing department of the company requires a minimum of 5,000 bbl/day of residual fuel, a new product. Residual fuel (RF) is straight run fuel oil (SRFO) directly from the atmospheric distillation column. The price is $10.00 /bbl, and it is sold "as is". Give the modifications required to the matrix in Figure 3-8 to determine the optimum way to operate the refinery with this new product.

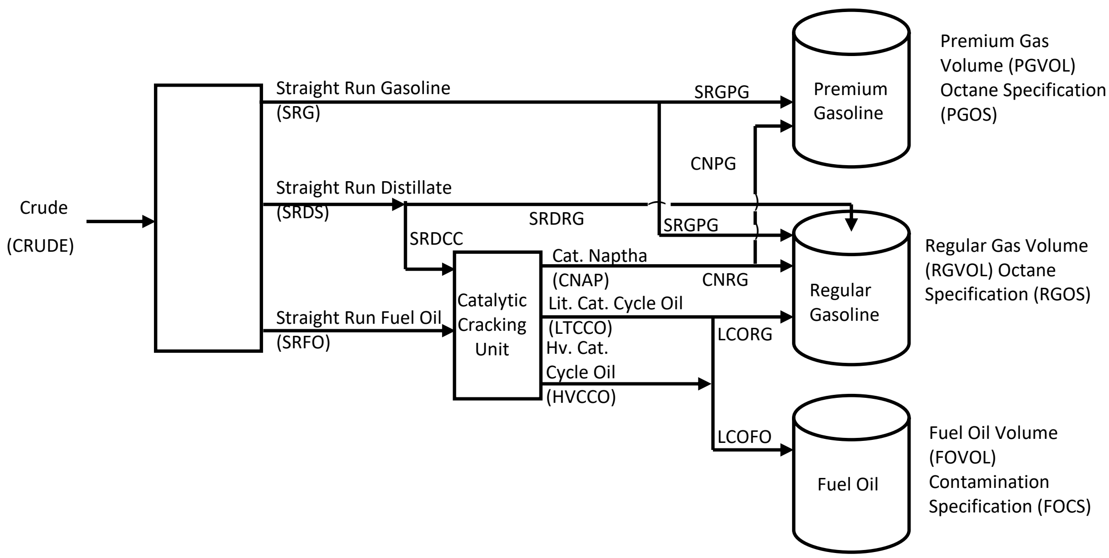

Figure 3-11 Flow Diagram of a Simple Refinery (8) in Problem 4-19

3-21. Prepare a matrix of the objective function and constraint equations from the process flow diagram for the contact process for sulfuric acid like the one given in Figure 3-8 for the simple refinery. The process flow diagram for the contact process is given in Figure 9-21. Use the following data, and assume that the units not included below have a fixed operating cost that do not affect the optimization.

Sales Prices and Raw Material Cost	*($/lb)*
Steam from Boiler 1 (STB1)	0.012
Steam from Boiler 2 (STB2)	0.012
Sulfuric Acid (H2SO4)	0.050
Sulfur to Burner (SULFUR)	0.025
Water to Economizer (WATER)	0.006
Make-up Water (MWATER)	0.006

Operating Costs	*($/lb)*
Steam from Boiler 1 (STB1)	0.001
Steam from Boiler 2 (STB2)	0.001
Air through Dryer (DRYAIR)	0.005
Water to Economizer (WATER)	0.001
Acid through acid cooler (H2SO4)	0.001
Acid through absorber (H2SO4)	0.001

Product Requirements and Raw Material Availability	*(lb/hr)*
Sulfuric Acid (H2SO4)	30,000
Steam (STB1 + STB2)	40,000
Sulfur (SULFUR)	10,000

Process Unit Maximum Capacities	*(lb/hr)*
Waste Heat Boiler 1(STB1)	25,000
Waste Heat Boiler 2(STB2)	25,000
Acid Cooler (H2SO4)	35,000
Dryer (DRYAIR)	150,000
Economizer (WATER)	60,000
Absorber (H2SO4)	35,000

Stream Split

Sulfuric Acid Production	= 3.06 SULFUR
Dry air	= 0.155 SULFUR
Make-up Water	= 0.128 SULFUR
Steam from Boilers 1 and 2	= WATER

3-22.[17] In linear programming there is a dual problem that is obtained from the original or primal problem. Many times, the dual problem can be solved with less difficulty than the primal one. The primal problem and corresponding dual problem are stated below in a general form.

Primal Problem		Dual Problem	
Maximize:	$\mathbf{c}^T\mathbf{x}$	Minimize:	$\mathbf{b}^T\mathbf{v}$
Subject to:	$\mathbf{A}\,\mathbf{x} \leq \mathbf{b}$	Subject to:	$\mathbf{A}^T\,\mathbf{v} \geq \mathbf{c}$
	$\mathbf{x} \geq \mathbf{0}$		$\mathbf{v} \geq \mathbf{0}$

The relationships between the primal and dual problems are summarized as follows. First, the dual of the dual is the primal problem. An m x n primal gives a n x m dual. For each primal constraint there is a dual variable and vice versa. For each primal variable there is a dual constraint and vice versa. The numerical value of the maximum of the primal is equal to the numerical value of the minimum of the dual. The solution of the dual problem is the Lagrange multipliers of the primal problem.

a. Give the primal problem of the following dual problem.

Minimize: $10v_1 + 15v_2$

Subject to: $v_1 + 5v_2 \geq 8$

$v_1 + v_2 \geq 4$

b. Solve the dual problem by the Simplex Method.

c. Using the solution of the dual problem, determine the optimal values for the variables in the primal problem.

3-23. The dual problem of linear programming can be obtained from the primal problem using Lagrange multipliers. Using the form of the equations given in Problem 4-22 for the primal problem and considering the slack variables have been added to the constraints, show that the Lagrange function can be written as:

$$L(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{c}^T \mathbf{x} + \boldsymbol{\lambda}^T (\mathbf{A}\,\mathbf{x} - \mathbf{b})$$

Rearrange this equation to give the following form.

$$L(\mathbf{x}, \boldsymbol{\lambda}) = -\mathbf{b}^T \boldsymbol{\lambda} + \mathbf{x}^T(\mathbf{A}^T \boldsymbol{\lambda} + \mathbf{c})$$

Justify that the following constrained optimization problem can be obtained from the Lagrange function:

Minimize: $\mathbf{b}^T \boldsymbol{\lambda}$

Subject to: $\mathbf{A}^T \boldsymbol{\lambda} \geq \mathbf{c}$

This is the dual problem given in Problem 3-22. Note that the independent variables of the dual problem are the Lagrange multipliers or "shadow prices" of the primal problem.

3-24. A primal programming can be converted into a dual problem as described in Problems 4-22 and 4-23. This approach is used when the dual problem is easier to solve than the primal problem. The general form of the primal problem and its dual was given in Problem 4-22.

a. Solve the dual problem of the primal problem and its dual given below.

Primal problem:
Minimize: $10x_1 + 6x_2 + 8x_3$
Subject to: $x_1 + x_2 + 2x_3 \geq 2$
$5x_1 + 3x_2 + 2x_3 \geq 1$

Dual problem:
Maximize: $2v_1 + v_2$
Subject to: $v_1 + 5v_2 \leq 10$
$v_1 + 3v_2 \leq 6$
$2v_1 + 2v_2 \leq 8$

b. In this procedure the solution of the primal problem is the negative of the coefficients of the slack variables in the objective function of the final iteration of the Simplex Method of the dual problem, and the solution of the dual problem is the negative of the Lagrange multipliers for the primal problem. Give the solution of the primal problem and the Lagrange multipliers for the primal problem and show that the minimum of the objective function of the primal problem is equal to the maximum of the objective function of the dual problem.

c. In the primal problem give the matrix to be inverted to compute the inverse of the optimal basis.

d. Compute the Lagrange multipliers using Equation 4-22 and show that they agree with the solution from the dual problem.

e. A new variable x_6 is added to the problem, as shown below.

$$\text{Minimize:} \quad 10x_1 + 6x_2 + 8x_3 + 2x_6 = p$$

$$\text{Subject to:} \quad x_1 + x_2 + 2x_3 + x_4 + 5x_6 = 2$$

$$5x_1 + 3x_2 + 2x_3 + x_5 + 3x_6 = 1$$

Will the optimal solution remain optimal or will the problem have to be resolved? Explain.

3-25. Solve Example 3-5 by the Two-Phase Method. In this method, the objective function is replaced by the sum of the artificial variables as a "new" objective function to be minimized. Then the Simplex Method is performed. The artificial variables will not be in the optimal solution since the minimum will have them be zero. First, the artificial variables are eliminated in the objective function to have the proper format to apply the Simplex Method with the artificial variables being the initially feasible basis. With each application of the Simplex Method an artificial variable is replaced in the basis by another variable, and the minimum is reached when all of the artificial variables have left the basis and are zero. At this point, the "new" objective function is replaced with the original objective function and the artificial variables are discarded. The Simplex Method is applied with the feasible basis obtained from the last step with the "new" objective function, and the algorithm is applied to reach the optimum.